CHAPTER 3

GRAPHING AND MAXIMUM-MINIMUM PROBLEMS

3.1 Continuity and the Intermediate Value Theorem

PREREQUISITES

1. Recall the definition of continuity (Section 1.2).
2. Recall how to compute limits (Section 1.2).

PREREQUISITE QUIZ

1. What two conditions must be met for a function f to be continuous at x_0 ?
2. Find the following limits:
   (a) $\lim_{x \to 2} x^3$
   (b) $\lim_{x \to 0} [(x^2 + 5x)/x]$
   (c) $\lim_{x \to 4} \sqrt{x^2 + 9}$

GOALS

1. Be able to describe the relationship between continuity and differentiability.
2. Be able to explain the intermediate value theorem in laymen's terms.

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STUDY HINTS

1. Points of continuity. By studying Example 1, you should notice that a function may be continuous at a point or on an interval which is composed of a set of points, at each of which the function is continuous.

2. Continuity at endpoints. The definition of continuity at a point given on p. 63 in the text needs a slight modification at endpoints. The reason is that two-sided limits do not exist at endpoints. This problem is dealt with by using one-sided limits when speaking about continuity at endpoints. (See p. 65).

3. Continuity tests. Differentiability at a point implies continuity at that same point. Rational functions are differentiable and therefore, are continuous, except where the denominator becomes zero. Continuity alone does not imply differentiability.

4. Intermediate value theorems. Notice that both versions require continuity. The first version simply says that to get from one side of $y = c$ to the other side, one must cross it. The second version says that if one doesn't cross $y = c$, one stays on the side one started on.

5. Method of bisection. Study Example 7 well; the technique will be very useful in the future for computing numerical answers. The interval does not always have to be halved; it is only a convenient way to converge upon the answer.

SOLUTIONS TO EVERY OTHER EXERCISE

1. (a) The function jumps at $x_0 = \pm 1$, the points of discontinuity. Thus, it is continuous on $(-\infty,-1)$, $(-1,1)$, and $(1,+\infty)$.

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1. (b) Even though the function is not differentiable at $x_0 = 2$, $\lim_{x\to 2} f(x) = f(2)$, so the function is continuous for all x.

   (c) The limit does not exist at $x_0 = -2$, so the function is continuous on $(-\infty,-1)$ and $(-1,+\infty)$.

5. At $x_0 = 1$, the denominator is $4 \neq 0$, so the function is continuous by the rational function rule.

9. $f(x)$ vanishes only at $x = \pm 1$. Thus, by the rational function rule, $f(x)$ is continuous on $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$, and in particular, $[-1/2,1/2]$.

13. This exercise is analogous to Example 4. We must show that $\lim_{x\to x_0} (f + g)(x) = (f + g)(x_0)$. By the sum rule for limits, $\lim_{x\to x_0} [f(x) + g(x)] = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = f(x_0) + g(x_0)$, since f and g are continuous at x_0. In addition, $(f + g)(x_0) = f(x_0) + g(x_0)$, so $\lim_{x\to x_0} (f + g)(x) = (f + g)(x_0)$ as required.

17. We must show that there exists s such that $f(s) = -s^5 + s^2 - 2s + 6 = 0$. Now $f(2) = -26 < 0$ and $f(-2) = 46 > 0$, so by the intermediate value theorem (first version), we conclude that there is some s_0 in $(-2,2)$ satisfying $f(s_0) = 0$. (Other intervals also work, for example, $[0,2]$.)

21. $f(-3) = -28$, $f(0) = 2$, $f(1.3) = -0.093$, and $f(2) = 2$. Combining this information with the fact that $f(\sqrt{2}) = f(-\sqrt{2}) = f(1) = 0$ and the second version of the intermediate value theorem, we conclude that $f(x)$ is negative on $(-\infty,-\sqrt{2})$ and $(1,\sqrt{2})$. $f(x)$ is positive on $(-\sqrt{2},1)$ and $(\sqrt{2},+\infty)$.

25. By the intermediate value theorem (second version), we know that $f(x) - 2 < 0$ on $[-1,1]$ because $f(0) - 2 = -2$. From $f(x) - 2 < 0$, we conclude $f(x) < 2$ on $[-1,1]$.

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29. In order for $f(x)$ to be continuous, we must have $\lim_{x\to 2} f(x) = f(2)$. $\lim_{x\to 2}[(x^2 - 4)/(x - 2)] = \lim_{x\to 2} (x + 2) = 4$, so define $f(2)$ as 4.

33. In order for $f(x)$ to be continuous, we must have $f(1) = \lim_{x\to 1} f(x)$. For $x \neq 1$, $f(x) = (x^2 - 1)/(x - 1) = [(x - 1)(x + 1)]/(x - 1) = x + 1$. Since $\lim_{x\to 1} f(x) = \lim_{x\to 1} (x + 1) = 2$, we define $f(1)$ as 2.

37. The function $f(x) = 1/(x - 1)$ is not continuous on the interval $[0,2]$, since it is not defined at $x = 1$; therefore, the intermediate value theorem does not apply to f and there is no contradiction.

41. Let x_0 be a root of $f(x) = x^5 + x^2 + 1$. The method of bisection, discussed in Example 7, is employed to develop the following chart.

Midpoint of I	Nature of f(midpoint)	Conclude that x_0 lies in I
	$f(-2) < 0 < f(-1)$	$(-2, -1)$
-1.5	$f(-1.5) < 0 < f(-1)$	$(-1.5, -1)$
-1.25	$f(-1.25) < 0 < f(-1)$	$(-1.25, -1)$
-1.125	$f(-1.125) > 0 > f(-1.25)$	$(-1.25, -1.125)$
-1.1875	$f(-1.1875) > 0 > f(-1.25)$	$(-1.25, -1.1875)$
-1.21875	$f(-1.21875) < 0 < f(-1.1875)$	$(-1.21875, -1.1875)$
-1.203125	$f(-1.203125) < 0 < f(-1.1875)$	$(-1.203125, -1.1875)$
-1.1953125	$f(-1.1953125) < 0 < f(-1.1875)$	$(-1.1953125, -1.1875)$
-1.1914062	$f(-1.1914062) > 0 > f(-1.1953125)$	$(-1.1953125, -1.1914062)$
-1.1933593	$f(-1.1933593) > 0 > f(-1.1953125)$	$(-1.1953125, -1.1933593)$
-1.1943359	$f(-1.1943359) < 0 < f(-1.1933593)$	$(-1.1943359, -1.1933593)$

So -1.194 approximates x_0 to within 0.001.

45. It is simplest to describe an example. Consider approximating the root $\sqrt{7}$ of $f(x) = x^2 - 7$ to within 0.01. Method 1 (bisection method): division point is the midpoint (see the solution to Exercise 24). Method 2: division point is the x-intercept $\bar{x}$ of the line L through $(a,f(a))$ and $(b,f(b))$ where $\bar{x}$ approximates the root and the nature of $f(\bar{x})$ determines the next interval (a,b) in which the root can be found.

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45 (continued).

Start with (a,b) = (2,3) as in Exercise 24. f(2) and f(3) have opposite signs, so Method 2 can be used. $\sqrt{7} \approx 2.6457513$.

n	(a,f(a))	(b,f(b))	L	$\bar{x}$	Accuracy	Nature of $f(\bar{x})$	(a,b)
1	(2,-3)	(3,2)	y = 5x - 13	2.6	0.1	$f(\bar{x}) < 0 < f(3)$	(2.6,3)
2	(2.6,-0.24)	(3,2)	y = (5.6)x - 14.8	2.6428571	0.01	$f(\bar{x}) < 0 < f(3)$	(2.6428571,3)

Hence we see that Method 2 requires only $n = 2$ steps while Method 1 requires $n = 7$ steps to attain the same accuracy. Now consider the method of dividing the interval(s), at each step, into 10 (instead of 2) equal parts and looking for sign changes at the 9 interior division points (instead of the midpoint). At the n^{th} step, there would be 10^n subintervals and so the size of the interval in which a solution can be found would be $1/10^n$ (assuming that the original interval has unit length). Hence n , the number of steps required to achieve the accuracy A , satisfies $1/10^n \leqslant A$. By letting A take the values $1/10$, $1/100$, ... , $1/10^n$, ... , we see that the number of steps required coincides with the number of digits of accuracy desired. Hence this method is more appropriate to the decimal system, but not necessarily the more efficient.

49. Let $p(x) = b_n x^n + \ldots + b_0$ with n = odd, $b_n \neq 0$. Then $f(x) = p(x)/b_n = x^n + a_{n-1}x^{n-1} + \ldots + a_0 (a_i = b_i/b_n$, $i = 0$, ... , $n - 1)$ has the same roots as $p(x)$. Hence it suffices to show that f has a root. $f(x)/x^n = 1 + a_{n-1}/x + \ldots + a_0/x^n \geqslant 1 - (1/|x|)\{|a_{n-1}| + |a_{n-2}|/|x| + \ldots + |a_0|/|x|^{n-1}\}$. For $|x| > 1$ and $|x| > 2\{|a_0| + \ldots + |a_{n-1}|\}$, $1/|x| < 1$ and $|x|/2 > \{|a_0| + \ldots + |a_{n-1}|\}$. So $(1/|x|)\{|a_{n-1}| + |a_{n-2}|/|x| + \ldots + |a_0|/|x|^{n-1}\} < (1/|x|)\{|a_{n-1}| + \ldots + |a_0|\} < (1/|x|)(|x|/2) = 1/2$. Hence $f(x)/x^n \geqslant 1 - (1/|x|)\{|a_{n-1}| +$

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49 (continued).

$|a_{n-2}|/|x| + \ldots + |a_0|/|x|^{n-1}\} > 1 - 1/2 > 0$. So $f(x)$ and x^n have the same sign, i.e., $f(x) < 0$ if x is large negative and $f(x) > 0$ if x is large positive since n is odd. By the intermediate value theorem, there is an x_0 with $f(x_0) = 0$.

SECTION QUIZ

1. (a) Is it true that all continuous functions are differentiable?

   (b) Is it true that all differentiable functions are continuous?

2. $f(x) = x^3 + 0.9x^2 - 0.75x - 0.25$ has three roots. Use the method of bisection or a similar technique to find them within 0.01 .

3. Let $g(x) = (x^2 - 4x - 5)/(x + 1)$. This is not defined at $x = -1$. How should $g(-1)$ be defined to maintain continuity?

4. True or false: If $f(-1) = -1$ and $f(1) = 1$, then $f(x)$ must be zero for some x in $(-1,1)$. Explain your answer.

5. Being employed by a maharajah can have its interesting moments. For example, just yesterday morning, one of the servants took one of the pet elephants out for a ride. Their position was given by $y = 5x$, until the elephant saw a mouse. Suddenly, at $x = 10$, their position became $y = -250 + 30x$.

   (a) Find the velocity function.

   (b) Is the position function continuous?

   (c) Is the velocity function continuous?

   (d) Does the intermediate value theorem tell you that somewhere in $(5,15)$, the velocity must be 20, which is between 5 and 30. Explain.

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SOLUTIONS TO PREREQUISITE QUIZ

1. $f(x_0)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$.

2. (a) 8

   (b) 5

   (c) 5

SOLUTIONS TO SECTION QUIZ

1. (a) No; consider $|x|$ when $x = 0$.

   (b) Yes

2. -1.32 , -0.27 , 0.70

3. -6

4. True if f is continuous; false if f is not continuous.

5. (a) $$v = \begin{cases} 5 & \text{if } x < 10 \\ 30 & \text{if } x > 10 \end{cases}$$

   (b) Yes

   (c) No

   (d) No; the velocity function is not continuous, so the intermediate value theorem does not apply.

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3.2 Increasing and Decreasing Functions

PREREQUISITES

1. Recall how to solve inequalities (Section R.1).
2. Recall the intermediate value theorem (Section 3.1).
3. Recall how to differentiate rational functions (Sections 1.4 and 1.5).

PREREQUISITE QUIZ

1. Find the solution set of $x^2 - 7x + 10 \geqslant 0$.
2. If $f(-2) = 3$ and $f(1) = -2$, and f is continuous, is it true that $f(x_0) = 0$ for some x_0 such that $-2 \leqslant x_0 \leqslant 1$? Cite a theorem which defends your answer.
3. Differentiate the following functions:
   (a) $2x^5 - 3x^2 + 2$
   (b) $(x + 8)(x^3 + 3x^2 + 2)$
   (c) $(x - 3)/(x^2 + 5x)$

GOALS

1. Be able to determine whether a function is increasing or decreasing by using the derivative.
2. Be able to classify critical points as a local minimum, a local maximum, or neither.

STUDY HINTS

1. Definition of increasing. Fig. 3.2.2 shows the graph of an increasing function which is also changing signs. By looking at the graph, the definition of increasing should become intuitively obvious. As you move from left to right, x is increasing and $f(x)$ is increasing

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1 (continued).

("getting larger") as well in a small interval (a,b) .

Note that the definition of increasing may not apply if the interval is chosen to be too large. For example, if we chose $[a,x_3]$ in Fig. 3.2.4, the function decreases at the right.

2. Changing signs. It should be intuitively obvious that an increasing function passing through $y = 0$ is changing from negative to positive. (See Fig. 3.2.2.) Similar reasoning applies to decreasing functions passsing through $y = 0$. (See Fig. 3.2.3.)

3. Increasing-decreasing test. Know this test well! A positive derivative means f is increasing; a negative derivative means f is decreasing. A zero derivative means that the slope of the graph is flat. Test you understanding by explaining all of the answers in Example 8.

4. Increasing on intervals. One can conclude that a function is increasing on an entire interval provided the derivative is positive throughout that interval. A similar statement may be made for the decreasing case. By using the intermediate value theorem, one can conclude that increasing and decreasing intervals are separated by points where $f'(x) = 0$.

5. Critical point test. The test is very important to know. The example $y = x^3$ is a common one to use when asked for a counterexample to demonstrate when the test fails. For $y = x^3$, $x_0 = 0$ is a critical point, but zero is neither a minimum nor a maximum. Also, note that the test only applies to differentiable functions. Think about $f(x) = |x|$.

6. First derivative test. This test should be understood, not memorized. Think about what the signs of the derivative mean and be able to reproduce Fig. 3.2.17.

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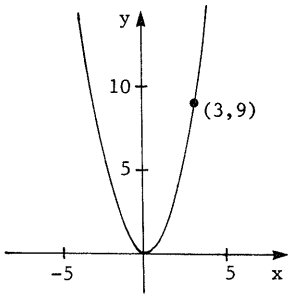
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Use the definitions of increasing and decreasing given at the beginning of this section. From the graph, we see that (a,b) may be chosen to be $(2,4)$. If $2 < x < 3$, then $f(x) < 9 = f(x_0)$. If $3 < x < 4$, then $f(x) > 9 = f(x_0)$. Thus, by definition, $f(x)$ is increasing at $x_0 = 3$.



5. For the general case, the chosen interval (a,b) must not be too large. In the case, (a,b) may be chosen as large as desired; we choose $(0,1)$. For $f(x) = 2x - 1$, $f(0) = -1 < 0$, $f(1/2) = 0$, and $f(1) = 1 > 0$, so f changes from negative to positive at $x_0 = 1/2$.

9. Using the increasing-decreasing test, we get $f'(x) = 3x^2 + 1$ and $f'(0) = 1$. Since $f'(x_0)$ is positive, f is increasing at $x_0 = 0$.

13. Here, $f'(t) = 5t^4 - 4t^3 + 4t$ and $f'(1) = 5$ is positive. Thus, $f(t)$ is increasing at $t = 1$, i.e., the particle is moving to the right.

17. Here, $f'(x) = 2x$, so $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$. Therefore, the increasing-decreasing test states that f is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

21. The derivative in (e) is a positive constant, so the function is an increasing linear function, namely, (5). The derivative in (b) is always positive, so the function is always increasing. Of the remaining functions, only (4) has this property. The derivative in (c) indicates that the function near $x = 0$ is increasing for $x < 0$ and decreasing for $x > 0$. Thus, (c) must match with (1). For (d), we look for a function

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21 (continued).

which decreases for <u>all</u> $x < 0$ and increases for <u>all</u> $x > 0$. Only (2) satisfies this. Note that (3) does not always increase for $x > 0$. It matches (a). Thus, the answer is (a) - (3), (b) - (4), (c) - (1), (d) - (2), (e) - (5).

25. We use the definitions of local minimum and local maximum on p. 151. The local minimum points occur at x_1, x_3, and x_5. Local maximum points are at x_2 and x_4. x_6 is neither a local minimum nor a local maximum point.

29. For $f(x) = x^3 + x^2 - 2$, $f'(x) = 3x^2 + 2x = x(3x + 2)$. $f'(x) = 0$ at $x = 0$ and $x = -2/3$, so these are the critical points. The sign of f' changes from negative to positive at $x = 0$ and from positive to negative at $x = -2/3$. Hence, the first derivative test states that 0 is a local minimum and $-2/3$ is a local maximum.

33. $\ell'(r) = 2(r^4 - r^2)(4r^3 - 2r) = 4r^3(r^2 - 1)(2r^2 - 1)$, so the critical points are 0, ± 1, $\pm 1/\sqrt{2}$. $\ell'(r)$ changes from negative to positive at $r = -1,0,1$. Therefore, these are local minima, by the first derivative test. $\ell'(r)$ changes from positive to negative at $\pm 1/\sqrt{2}$, the local maxima.

37. $f'(x) = m$. For $m = -2$, the sign change is from positive to negative since the function is decreasing. There is no sign change when $m = 0$. For $m = 2$, the sign change is from negative to positive since the function is increasing.

41. $f(x) = x^2 - 4x + 4$ has a double root at $x = 2$, so $f(2) = 0$. $f'(x) = 2x - 4$, which is negative for $x < 2$ and positive for $x > 2$. Thus, $f'(x)$ changes sign at 2; therefore, $x = 2$ is a local minimum. Consequently, f does not change sign anywhere.

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45. For $f(x) = 2x^3 - 5x + 7$, $f'(x) = 6x^2 - 5$. $f'(x)$ is positive for $|x| > \sqrt{5/6}$ and negative for $|x| < \sqrt{5/6}$. Hence f is increasing on $(-\infty,-\sqrt{5/6})$ and $(\sqrt{5/6},\infty)$; f is decreasing on $(-\sqrt{5/6},\sqrt{5/6})$.

49. $f(x) = ax^2 + bx + c$, so $f'(x) = 2ax + b$. The hypothesis that $f' < 0$ for $x < 2$ and $f' > 0$ for $x > 2$ implies $x = 2$ is a local minimum and hence a critical point, i.e., $0 = f'(2) = 4a + b$. The hypothesis that $x = 1$ is a zero of f implies $0 = f(1) = a + b + c$. Hence $b = -4a$, $c = -a - b = 3a$, and so $f(x) = ax^2 - 4ax + 3a = a(x^2 - 4x + 3)$ where $a > 0$ since a parabola with a minimum is a parabola which opens upward.

53. An increasing f and g at x_0 implies that $f(x) < f(x_0) < f(y)$ and $g(x) < g(x_0) < g(y)$ for $x < x_0 < y$. Hence, $f(x) + g(x) < f(x_0) + g(x_0) < f(y) + g(y)$ for $x < x_0 < y$, showing that $f + g$ is increasing at x_0.

57. (a) gh is increasing if $(gh)' = g'h + gh' > 0$. Divide both sides of the inequality by gh (which is positive and hence preserves the inequality) to yield the desired criterion $g'/g + h'/h > 0$. gh is decreasing if $g'/g + h'/h < 0$.

(b) g/h is increasing if $(g/h)' = (g'h - h'g)/h^2 > 0$. Multiplying through by (h/g) gives $g'/g - h'/h > 0$. g/h is decreasing if $g'/g - h'/h < 0$.

61. Since the graph does not change sign at $x = 0$, the quartic has the form $p(x) = cx^2(x + b)(x - b) = cx^4 - cb^2x^2$, where c is a non-zero constant. Thus, $p'(x) = 4cx^3 - 2cb^2x$, which is 0 at $x = 0,\pm a$. When $x = a$, $4ca^3 - 2cb^2a = 0$ or $2a^2 = b^2$. We obtain the same relationship when $x = -a$. At $x = b$, we need $p'(x) > 0$, so $c > 0$. Hence, the polynomial is $p(x) = cx^4 - cb^2x^2$, $c > 0$, and $2a^2 = b^2$.

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SECTION QUIZ

1. Consider the function $f(x) = x^3/(1 - x^2)$. On what intervals is f increasing? decreasing?
2. What are the critical points of $x^3/(1 - x^2)$? Classify any critical points as a local minimum, local maximum, or neither.
3. Sketch examples of the following types of critical points:
   (a) local minimum
   (b) local maximum
   (c) neither a local minimum nor a local maximum.
4. True or false: All local minima and local maxima are points where the derivative is zero.
5. Your house guest snores every evening with the same pattern. In hours from midnight, the volume is given by $S(t) = t^3 - 9t^2 + 15t$, $0 \leqslant t \leqslant 6$. If $S(t) < 0$, your house guest is not snoring.
   (a) What are the critical points of $S(t)$?
   (b) Classify the critical points as minima or maxima.
   (c) Where is $S(t)$ increasing? decreasing?

SOLUTIONS TO PREREQUISITE QUIZ

1. $x \leqslant 2$ or $x \geqslant 5$
2. Yes, use the intermediate value theorem.
3. (a) $10x^4 - 6x$
   (b) $(x^3 + 3x^2 + 2) + (x + 8)(3x^2 + 6x) = 4x^3 + 33x^2 + 48x + 2$
   (c) $(-x^2 + 6x + 15)/(x^2 + 5x)^2$

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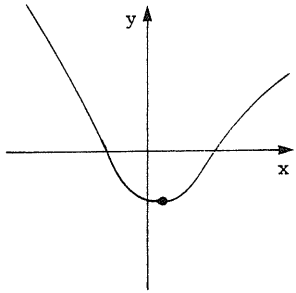
SOLUTIONS TO SECTION QUIZ

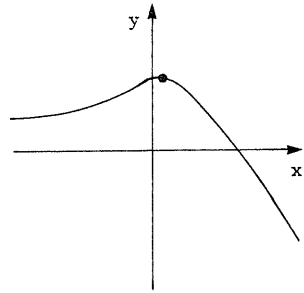
1. Increasing on $(-\sqrt{3},-1)$, $(-1,1)$, and $(1,\sqrt{3})$; decreasing on $(-\infty,-\sqrt{3})$, $(\sqrt{3},\infty)$.

2. $x = -\sqrt{3}$: local minimum; $x = 0$: neither; $x = \sqrt{3}$: local maximum.

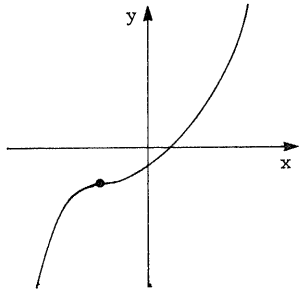
3. (a)



(b)



(c)



4. False; consider $f(x) = |x|$ at $x = 0$.

5. (a) $t = 1,5$

(b) $t = 1$: maximum; $t = 5$: minimum.

(c) Increasing on $(0,1)$, $(5,6)$; decreasing on $(1,5)$.

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3.3 The Second Derivative and Concavity

PREREQUISITES

1. Recall how derivatives can be used to show that a function is increasing or decreasing (Section 3.2).
2. Recall how to compute a second derivative (Section 2.1).

PREREQUISITE QUIZ

1. Let $g(t) = t^3 + 3t^2$. On what intervals is $g(t)$ increasing?
2. Let $f(x) = x/(x + 2)$. On what intervals is $f(x)$ increasing? Decreasing?
3. Find the second derivatives of the functions given in Questions 1 and 2.

GOALS

1. Be able to use the second derivative to determine the concavity of a function.
2. Be able to find the inflection points of a function.

STUDY HINTS

1. Second derivative test. There is a simple way to remember that $f''(x_0) > 0$ and $f'(x_0) = 0$ implies a local minimum. Since $f'' > 0$, f' must be increasing. And since $f'(x_0) = 0$, $f'(x) < 0$ on the left of x_0 and $f'(x) > 0$ on the right side. Thus $f(x)$ is decreasing to a local minimum at x_0, where it begins to increase again. Deriving the second derivative test in this fashion may be easier than memorizing it.
2. Concavity. A curve is concave upward if it can "hold water." In order for this to occur, the slopes must be getting larger, i.e., $f'(x)$ is increasing or $f''(x) \geqslant 0$. Similarly, $f''(x) \leqslant 0$ implies downward concavity, and these curves "spill water."

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3. Concave up on intervals. As with the concept of increasing and decreasing, an entire interval may possess a certain concavity. By using the intermediate value theorem, one can conclude that these intervals are separated by points x where $f''(x) = 0$.

4. Inflection points. Concavity changes at those points where $f''(x) = 0$ and $f'''(x) \neq 0$. Notice the similarity of this definition with that of minimum and maximum points, where $f'(x)$ changes signs.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $f'(x) = 6x$, which is 0 at $x = 0$; $f''(x) = 6$ is positive, so by the second derivative test, $x = 0$ is a local minimum.

5. $f'(x) = 4x/(x^2 + 1)^2$, which is 0 at $x = 0$; $f''(x) = (4 - 12x^2)/(x^2 + 1)^3$. $f''(0) = 4$ which is positive, so by the second derivative test, $x = 0$ is a local minimum.

9. Since $f'(x) = 6x + 8$, $f''(x) = 6$ which is positive for all x . Thus, $f(x)$ is concave upward everywhere.

13. Since $f'(x) = -1/(x - 1)^2$, $f''(x) = 2/(x - 1)^3$. The second derivative is positive for $x > 1$ and negative for $x < 1$. Thus, $f(x)$ is concave upward on $(1,\infty)$ and concave downward on $(-\infty,1)$.

17. For $f(x) = x^3 - x$, $f'(x) = 3x^2 - 1$, $f''(x) = 6x$, and $f'''(x) = 6$. The second derivative vanishes at $x = 0$ and since $f'''(0) \neq 0$, it is an inflection point.

21. $f'(x) = 4(x - 1)^3$; $f''(x) = 12(x - 1)^2 = 0$ at $x = 1$. Note that $f''(x) \geqslant 0$ for all x , so it does not change sign; therefore, there are no inflection points.

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25. (a) x_0 is a local maximum point since $f'(x)$ changes from positive to negative.

(b) x_0 is an inflection point since $f(x)$ changes from concave downward to concave upward.

(c) x_0 is neither. $f'(x)$ is positive on both sides of x_0 and $f''(x) = 0$ on the left of x_0, so $f''(x)$ does not change sign.

(d) x_0 is a local maximum point since $f'(x)$ changes from positive to negative.

(e) x_0 is a local maximum point since $f'(x)$ changes from positive to negative.

(f) x_0 is an inflection point since $f(x)$ changes from concave upward to concave downward.

(g) x_0 is an inflection point since $f(x)$ changes from concave downward to concave upward.

(h) x_0 is a local minimum point since $f'(x)$ changes from negative to positive.

29. $f(x) = x^3 + 2x^2 - 4x + 3/2$; $f'(x) = 3x^2 + 4x - 4 = 0$ at $x = 2/3$, -2 . f' is negative between its roots and positive outside its roots. Hence f is decreasing on $(-2,2/3)$ and increasing on $(-\infty,-2)$ and $(2/3,\infty)$. $f''(x) = 6x + 4 = 0$ at $x = -2/3$; $f'''(-2/3) = 6 \neq 0$, so $x = -2/3$ is an inflection point. As $f''(x) < 0$ for $x < -2/3$ and $f''(x) > 0$ for $x > -2/3$, f is concave downward on $(-\infty,-2/3)$ and concave upward on $(-2/3,\infty)$. In particular, $f''(-2) < 0$ and $f''(2/3) > 0$, so $x = -2$ is a local maximum and $x = 2/3$ is a local minimum.

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33. f has inflection points at 1 and 2 if $f''(x) = k(x - 1)(x - 2)$, $k \neq 0$. Practice with differentiation tells us that if f'' is a quadratic, then f' is a cubic and f is a quartic. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, then $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$; $f''(x) = 12ax^2 + 6bx + 2c$ which should $= k(x^2 - 3x + 2)$. Hence $a = k/12$; $b = -k/2$; $c = k$; d, e are arbitrary. So $f(x) = kx^4/12 - kx^3/2 + kx^2 + dx + e$. A specific example is $f(x) = x^4/12 - x^3/2 + x^2 - x - 1$. (Alternatively, use the method of antidifferentiation in Section 2.5.)

37. (a) For x near x_0, the linear approximation of $f(x)$ at x_0 is the linear function $\ell(x) = f(x_0) + f'(x_0)(x - x_0)$. $f(x) = x^3 - x$; $f(-1) = f(0) = f(1) = 0$. $f'(x) = 3x^2 - 1$; $f'(\pm 1) = 2$; $f'(0) = -1$. Hence $\ell(x)$ at $x_0 = -1$, 0, 1 is $2(x + 1)$, $-x$, $2(x - 1)$, respectively.

(b)

x_0	$f''(x_0)$	Δx	$x = x_0 + \Delta x$	$f(x)$	$\ell(x)$	$e(x) = f(x) - \ell(x)$ error
-1	-6	1	0	0	2	-2
		-1	-2	-6	-2	-4
		0.1	-0.9	0.171	0.2	-0.029
		-0.1	-1.1	-0.231	-0.2	-0.031
		0.01	-0.99	0.019701	0.02	-0.000299
		-0.01	-1.01	-0.020301	-0.02	-0.000301
0	0	1	1	0	-1	1
		-1	-1	0	1	-1
		0.1	0.1	-0.099	-0.1	0.001
		-0.1	-0.1	0.099	0.1	-0.001
		0.01	0.01	-0.009999	-0.01	0.000001
		-0.01	-0.01	0.009999	0.01	-0.000001
1	6	1	2	6	2	4
		-1	0	0	-2	2
		0.1	1.1	0.231	0.2	0.031
		-0.1	0.9	-0.171	-0.2	0.029
		0.01	1.01	0.020301	0.02	0.000301
		-0.01	0.99	-0.019701	-0.02	0.000299

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37. (b) (continued)

The table shows that $e(x) < 0$ if $f''(x_0) < 0$ and $e(x) > 0$ if $f''(x_0) > 0$. If $f''(x_0) = 0$, $e(x)$ is comparatively smaller and decreases faster as Δx decreases. The sign of the error is the same as for Δx when $f''(x_0) = 0$.

41. $s'(t) = 40 - 32t$, which vanishes at $t = 5/4$; $s''(5/4) = -32 < 0$, so the maximum height is attained at $t = 5/4$ seconds. The maximum height is $s(5/4) = 28$ feet.

45. No. Counterexample: $f(x) = x^2 + 1 > 0$ for all x; $f''(x) = 2 \neq 0$ for all x, so f has no inflection points. Let $g(x) = 1/f(x) = 1/(x^2 + 1)$; $g'(x) = -2x/(x^2 + 1)^2$; $g''(x) = 2(3x^2 - 1)/(x^2 + 1)^3 = 0$ at $x = \pm\sqrt{3}/3$. As $x^2 + 1 > 0$, $(x^2 + 1)^3 > 0$, so $g''(x)$ changes sign wherever $3x^2 - 1$ does. Since $3x^2 - 1$ is a quadratic with two distinct roots, it changes sign at its roots $x = \pm\sqrt{3}/3$. Hence g has inflection points at $x = \pm\sqrt{3}/3$ while f has none.

SECTION QUIZ

1. Consider the function $f(x) = x^3/(1 - x^2)$. On what intervals is $f(x)$ concave upward? concave downward?
2. Where is $f''(x) = 0$ for $f(x) = x^3/(1 - x^2)$? Is this an inflection point? Why or why not?
3. True of false: Suppose f is twice differentiable. If $x = 3/2$ is the only inflection point of a continuous function f and $f''(x) < 0$ for $x < 3/2$, then $f''(x) \geqslant 0$ for $x > 3/2$. Explain.
4. Paranoid Pete saw some UFO's last night. Thinking that we are about to be invaded, he orders an underground shelter to be built. In order to fool the invaders, he wants an uneven terrain. The shape of the terrain

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4. (continued)

is described by $w = x^4/12 - x^3/5 + x^2/10 + 2x - 3$, $x \geq 0$. In addition, Paranoid Pete is very concerned about drainage in case of rain; therefore, he needs to know the concavity of the terrain.

(a) Where are the inflection points, if any?

(b) Discuss the concavity of the terrain.

ANSWERS TO PREREQUISITE QUIZ

1. $(-\infty,-2)$ and $(0,\infty)$

2. Increasing for $x \neq -2$; decreasing nowhere.

3. $g''(t) = 6t + 6$; $f''(x) = -4/(x + 2)^3$

ANSWERS TO SECTION QUIZ

1. Upward: $(-\infty,-1)$, $(0,1)$; downward: $(-1,0)$, $(1,\infty)$

2. 0 ; yes; $f''' \neq 0$

3. True; concavity changes at an inflection point. $f'' < 0$ implies downward concavity, so the graph must be concave upward for $x > 3/2$.

4. (a) $x = 1/5$, 1

(b) Concave downward: $[0,1/5)$, $(1,\infty)$; concave upward: $(1/5,1)$

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3.4 Drawing Graphs

PREREQUISITES

1. Recall how derivatives give information about local extrema, increasing, decreasing, concavity, and inflection points (Sections 3.2 and 3.3).
2. Recall how to compute limits at infinity (Section 1.2).
3. Recall how to compute infinite limits (Section 1.2).

PREREQUISITE QUIZ

1. If x_0 is a local extreme point, what can you say about the value of $f'(x_0)$?
2. If a function g is increasing at x_0, what can you say about $g'(x_0)$?
3. If $g'(t_0) > 0$, is g increasing or decreasing at t_0 ?
4. If $f''(x_0) < 0$, is f concave upward or downward at x_0 ?
5. Compute the following limits, stating $\pm\infty$, if appropriate:
   (a) $\lim_{x\to\infty} [(2x^2 - 1)/(x^2 + 1)]$
   (b) $\lim_{x\to-\infty} [(3x^3 + x - 1)/(x^3 + x)]$
   (c) $\lim_{x\to0+} (1/x)$
   (d) $\lim_{x\to2-} [1/(x - 2)]$

GOALS

1. Be able to use calculus for the purpose of sketching graphs.

STUDY HINTS

1. Symmetries. Before jumping into the problem, note any symmetries. If any symmetry exists, your work is reduced by half. The two important symmetries are those about the y-axis and about the origin. Note that

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1 (continued).

if all of the exponents in a polynomial are even, the function is even. Similarly, all odd exponents make the function odd. The last two statements apply only to polynomials. A quotient such as $(x^3 + x^5)/(x + x^3)$ has all odd exponents, but it is an even function. Remember that all constant terms have an even exponent, namely, zero.

2. Horizontal asymptotes. If limits exist at $\pm\infty$, these limits are the horizontal asymptotes. Each function can have at most two horizontal asymptotes. Note that a function may cross the asymptote one or more times as x increases; consider $f(x) = x/(1 + x^2)$ in Fig. 3.4.9 .

3. Vertical asymptotes. We are interested in the behavior on both sides of the x-values which produce a zero denominator. The sign of the limiting values on either side of a vertical asymptote may either be the same or opposite. Consider $f(x) = 1/x^2$ and $f(x) = 1/x$. Note that, unlike horizontal asymptotes, there may be any number of vertical asymptotes.

4. Other asymptotes. Example 3 is interesting in that one of the asymptotes is neither vertical nor horizontal.

5. Six-step method. There is no need to memorize the method. With enough practice, the method should become second nature to you.

6. Cusps. These occur when the limit of the derivative becomes infinite, and when the sign of the derivative changes. Note the difference when the sign does not change, by comparing Figs. 3.4.15 and 3.4.16 . You need to be aware of the possibility of cusps only if fractional powers occur, such as $f(x) = x^{2/3}$.

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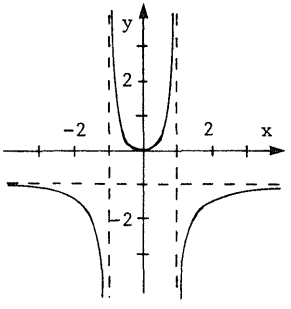
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. f is odd because $f(-x) = [(-x)^3 + 6(-x)]/[(-x)^2 + 1] = -(x^3 + 6x)/(x^2 + 1) = -f(x)$.

5. $f(x) = x/(x^3 + 1)$ is undefined at $x = -1$, so it is a vertical asymptote. For x near -1 and $x < -1$, x is negative and $x^3 + 1$ is small and negative, so $f(x)$ is large and positive. For x near -1 and $x > -1$, x is negative and $x^3 + 1$ is small and positive, so $f(x)$ is large and negative.

9. 

$f(-x) = f(x)$, so f is even and we need to consider $x \geqslant 0$ only. $x = 1$ is a vertical asymptote. For x near 1 and $x < 1$, $f(x)$ is large and positive. For x near 1 and $x > 1$, $f(x)$ is large and negative. $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} [1/(1/x^2 - 1)] = -1$, which is a horizontal asymptote. $f'(x) = 2x/(1 - x^2)^2$, so 0 is a critical point and f is increasing on $(0,1)$ and $(1,\infty)$. $f''(x) = (2 + 6x^2)/(1 - x^2)^3$, so f'' has the same sign as $1 - x^2$. There are no inflection points. $f''(0) = 2 > 0$, so 2 is a local minimum. f is concave upward on $(0,1)$ and concave downward on $(1,\infty)$. Plot a few points and use the symmetry to complete the graph.

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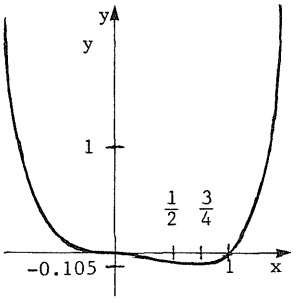
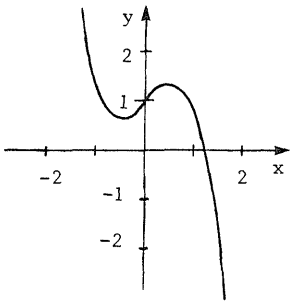
13.



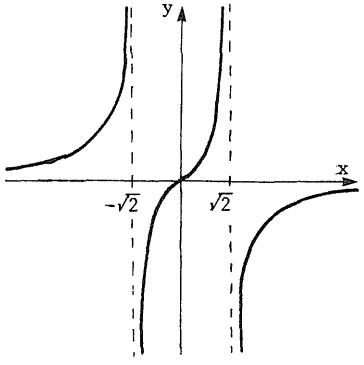
$f(x)$ shows no symmetry. $f(x) = x^3(x - 1) = 0$ at $x = 0,1$. $f'(x) = x^2(4x - 3) = 0$ at $x = 0,3/4$. $f''(x) = 6x(2x - 1) = 0$ at $x = 0,1/2$; $f'''(x) = 6(4x - 1)$; $f'''(0) \neq 0$ and $f'''(1/2) \neq 0$, so 0 and 1/2 are inflection points. $f''(3/4) > 0$, so 3/4 is a local minimum. $f''(x) > 0$ for $x > 1/2$ and for $x < 0$; $f''(x) < 0$ for $0 < x < 1/2$; so f is concave upward on $(1/2,\infty)$ and $(-\infty,0)$, concave downward on $(0,1/2)$.

17.



Even though $f(x)$ is not symmetric about the y-axis or the origin, note that $f(x)$ is the odd function $-x^3 + x$ shifted up 1 unit. $f'(x) = -3x^2 + 1 = 0$ at $x = \pm 1/\sqrt{3}$. $f''(x) = -6x = 0$ at $x = 0$. $f'''(x) = -6$, so $x = 0$ is an inflection point. $f''(-1/\sqrt{3}) > 0$, so $-1/\sqrt{3}$ is a local minimum and $(-\infty,0)$ is where f is concave up. $f''(1/\sqrt{3}) < 0$, so $1/\sqrt{3}$ is a local maximum and the graph is concave down on $(0,\infty)$.

21.



$f(x) = [x^3/(2 - x^2)] + x = [x^3 + x(2 - x^2)]/(2 - x^2) = 2x/(2 - x^2)$ is odd since $f(-x) = -f(x)$. Hence we only need to graph f for $x \geqslant 0$ (the graph for $x \leqslant 0$ can be obtained from that for $x \geqslant 0$ by reflecting the latter across the x-axis and then across the y-axis, or the y-axis first and then the x-axis). f has vertical asymptotes at

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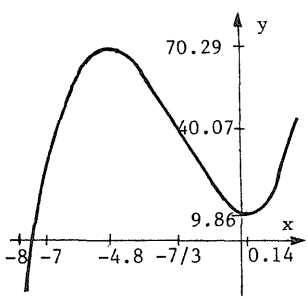
21 (continued).

$x = \pm\sqrt{2}$; for x near $\sqrt{2}$ and $x < \sqrt{2}$, $f(x)$ is large and positive for x near $\sqrt{2}$ and $x > \sqrt{2}$, $f(x)$ is large and negative. As $|x|$ gets large, $f(x) = (2x/x^2)/[(2 - x^2)/x^2] = (2/x)/[(2/x^2) - 1]$ approaches 0 , i.e., the line $y = 0$ is a horizontal asymptote. $x = 0$ is the only zero of f . $f'(x) = 2(2 + x^2)/(2 - x^2)^2$ has no real zeros. Hence f has no critical point. $f''(x) = [4x(x^2 + 6)]/(2 - x^2)^3 = [2(x^2 + 6)/(2 - x^2)^2]f(x)$; so f'' has the same sign as f , i.e., $f'' > 0$ on $(-\infty,-\sqrt{2})$ and $(0,\sqrt{2})$, $f'' < 0$ on $(-\sqrt{2},0)$ and $(\sqrt{2},\infty)$, which implies f is concave upward on $(-\infty,-\sqrt{2})$ and $(0,\sqrt{2})$, concave downward on $(-\sqrt{2},0)$ and $(\sqrt{2},\infty)$. $f'''(x) = 12(x^4 + 12x^2 + 4)/(2 - x^2)^4$. Hence $f''(0) = 0$ but $f'''(0) \neq 0$, so $x = 0$ is an inflection point.

25.



$f(x) = x^3 + 7x^2 - 2x + 10$ is neither odd nor even, and is defined everywhere. $f'(x) = 3x^2 + 14x - 2 = 0$ at $x = (-7 \pm \sqrt{55})/3 \approx 0.14, -4.8$, respectively. f' is negative between its roots and positive outside its roots, so f is decreasing on $(-4.8,0.14)$, increasing on $(-\infty,-4.8)$ and $(0.14,\infty)$. f' changes sign (from positive to negative) at -4.8 and (from negative to positive) at 0.14 , so $x = -4.8$ is a local maximum with maximum value $f(-4.8) \approx 70.29$ and $x = 0.14$ is a local minimum with minimum value $f(0.14) \approx 9.86$. $f''(x) = 6x + 14 = 0$ at $x = -7/3$; $f'''(-7/3) = 6 \neq 0$, so $-7/3$ is an inflection point.

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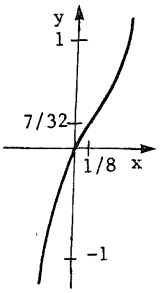
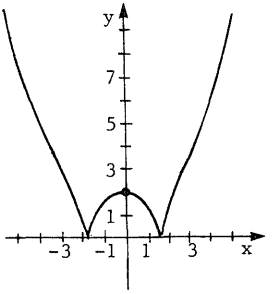
29.



$f(x) = 8x^3 - 3x^2 + 2x = x(8x^2 - 3x + 2)$ is defined everywhere, is neither odd nor even, and has only one real zero (at $x = 0$). $f'(x) = 2(12x^2 - 3x + 1)$ has no zero. $f''(x) = 48x - 6 = 0$ at $x = 1/8$, $f'''(1/8) = 48 \neq 0$, so $1/8$ is an inflection point with $f(1/8) = 7/32$. $f''(x) > 0$ for $x > 1/8$ and $f''(x) < 0$ for $x < 1/8$. Hence f is concave upward on $(1/8,\infty)$ and concave downward on $(-\infty,1/8)$.

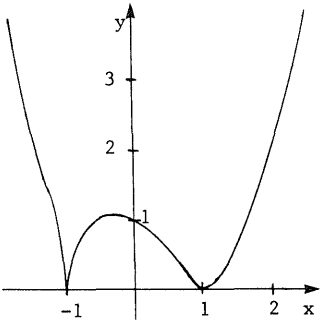
33. (a) $f'(x) = (2/3)(x^2 - 3)^{-1/3}(2x) = 4x/3(x^2 - 3)^{1/3}$. $f'(x) = 0$ only at $x = 0$, and $f'(x)$ is undefined at $x = \pm\sqrt{3}$. $f'(x) > 0$ on $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$, the intervals where f is increasing. On $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$, $f'(x) < 0$ and f is decreasing.

(b)



$f(x)$ is an even function. This is a nonnegative function with zeros at $\pm\sqrt{3}$. $f''(x) = 4(x^2 - 9)/9(x^2 - 3)^{4/3}$, so inflection points occur at $x = \pm 3$. Cusps occur at $x = \pm\sqrt{3}$.

37.



$f(x) = (x - 1)^{4/3}(x + 1)^{2/3}$; $f'(x) = (4/3)(x - 1)^{1/3}(x + 1)^{2/3} + (2/3)(x - 1)^{4/3} \times (x + 1)^{-1/3} = (x - 1)^{1/3}(x + 1)^{-1/3}(4x + 4 + 2x - 2)/3 = (2/3)(x - 1)^{1/3}(x + 1)^{-1/3}(3x + 1)$. $f'(x) = 0$ if $x = 1, -1/3$. $\lim_{x \to -1}[1/f'(x)] = 0$, so a cusp exists at $x = -1$. $f''(x) = (2/3)(x + 1)^{-1/3}[(1/3)(x - 1)^{-2/3}(3x - 1) + (x - 1)^{1/3} \cdot 3] + (2/3)(-1/3)(x + 1)^{-4/3}(x - 1)^{1/3} \times$

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37 (continued).

$(3x-1) = (2/9)(x+1)^{-4/3}(x-1)^{-2/3}(15x^2 + 10x - 13)$. $f''(x) = 0$ if $15x^2 + 10x - 13 = 0$, i.e., $x = (-10 \pm \sqrt{880})/30 \approx 0.655, -1.322$; $f''(x)$ is not defined at $x = \pm 1$. f is increasing on $(-1,-1/3),(1,\infty)$, and decreasing on $(-\infty,-1),(-1/3,1)$. The graph is concave upward on $(-\infty,-1.322),(0.655,\infty)$, and concave downward on $(-1.322,0.655)$.

41. f is even if and only if $f(x) = f(-x)$, i.e., $f(x) - f(-x) = 0$ for all x. For $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $f(-x) = a_n(-x)^n + a_{n-1}(-x)^{n-1} + \ldots + a_1(-x) + a_0$. Thus, $f(x) - f(-x) = 2a_1 x + 2a_3 x^3 + \ldots = 0$ for all x if and only if $a_1 = a_3 = a_5 = \ldots = 0$, i.e., f is even if and only if the even powers of x occur with nonzero coefficients in $f(x)$. The case where $f(x)$ is odd is done in a similar fashion.

45. $f(x) = e(x) + o(x)$, $f(-x) = e(-x) + o(-x) = e(x) - o(x)$ since e is even and o is odd. Adding and subtracting these two equations yield $f(x) + f(-x) = 2e(x)$, $f(x) - f(-x) = 2o(x)$; so $e(x) = [f(x) + f(-x)]/2$ and $o(x) = [f(x) - f(-x)]/2$. We check that $[f(x) + f(-x)]/2$ and $[f(x) - f(-x)]/2$ are indeed even and odd, respectively, and that their sum is $f(x)$.

49. $f(x) = ax^3 + bx^2 + cx + d$ $(a \neq 0)$; $f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b = 0$ at $x = -b/3a$; $f'''(-b/3a) = 6a \neq 0$, so f has an inflection point at $x = -b/3a$ and nowhere else. $g(x) = f(x - b/3a) - f(-b/3a) = a(x - b/3a)^3 + b(x - b/3a)^2 + c(x - b/3a) + d - a(-b/3a)^3 - b(-b/3a)^2 - c(-b/3a) - d = ax^3 + (c - b^2/3a)x$ is odd and $g(0) = 0$ (two properties which g would not have possessed had g been not so defined as to eliminate the quadratic and the constant terms).

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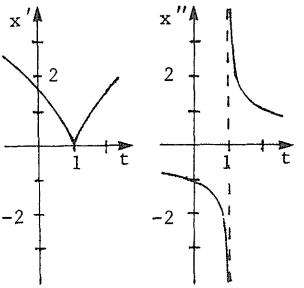
49 (continued).

$g'(x) = 3ax^2 + (c - b^2/3a)$; $g''(x) = 6ax = 0$ only at $x = 0$. $g'''(0) = 6a \neq 0$, so g has an inflection point at $x = 0$ and nowhere else. Hence the point $(0,g(0)) = (0,0)$ acts for g as $(-b/3a,f(-b/3a))$ did for f . As f and g are cubics with the same leading coefficients, their graphs have the same basic shape. Hence f is symmetric about its inflection point $(-b/3a,f(-b/3a))$ provided g is symmetric about its inflection point $(0,g(0)) = (0,0)$ which is the case since g is odd.

53. 

The velocity is given by $x'(t) = (5/3) \times (t - 1)^{2/3}$ so $\lim_{t \to 1} [x'(t)] = 0$. The acceleration is $x''(t) = (10/9)(t - 1)^{-1/3}$, so $\lim_{t \to 1} [x''(t)]$ is infinite. See the graphs of $x'(t)$ and $x''(t)$ at the left. The graph of $x(t)$ is shown in the answer section of the text.

57. $f(x - b/4a) = a(x - b/4a)^4 + b(x - b/4a)^3 + c(x - b/4a)^2 + d(x - b/4a) + e = a(x^4 - bx^3/a + 3b^2x^2/4a^2 - b^3/16a^3 + b^4/256a^4) + b(x^3 - bx^2/4a + bx/16a^2 - b^3/64a^3) + c(x^2 - bx/2a + b^2/16a^2) + d(x - b/4a) + e = ax^4 + (-b + b)x^3 + (3b^2/4a - b^2/4a + c)x^2 + (-b^2/16a^2 + b^2/16a^2 + cb/2a + d)x + (b^4/256a^3 - b^4/64a^3 + cb^2/16a^2 + db/4a + e)$. Dividing by a and using new coefficients, we have $f(x) = x^4 + c_1x^2 + d_1x + e$.

61. A type I quartic occurs in the right half-plane where $c > 0$. A type II quartic occurs on the d-axis. Type II_1 occurs on the positive d-axis; type II_2 occurs at the origin, and type II_3 occurs on the negative d-axis. A type III quartic occurs in the left half-plane where $c < 0$. Type III_1 occurs in the upper white region where $d > -(4c/3)\sqrt{-c/6}$; type III_2 occurs on the curve which forms the upper boundary of the gray region

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61 (continued).

where $d = -(4c/3)\sqrt{-c/6}$; type III_3 occurs in the gray region where $|d| < |(4c/3)\sqrt{-c/6}|$; type III_4 occurs on the curve which forms the lower boundary of the gray region where $d = (4c/3)\sqrt{-c/6}$, and type III_5 occurs in the lower white region where $d < (4c/3)\sqrt{-c/6}$.

SECTION QUIZ

1. Consider the function $f(x) = x^3/(1 - x^2)$

   (a) Discuss the symmetries of this function.

   (b) Where are the asymptotes of this function?

   (c) Sketch the graph.

2. Sketch the graph of $f(x) = (1 - x^{2/3})^3$. Clearly label the critical points, the inflection points, and the cusps.

3. A brilliant piratess had left a treasure map for her husband. Clues describing the exact location of a golden treasure were left along the curves described by $y = (x^3 - 1)/(x^3 - 3)$. The origin is located at the center of their island home and each unit represents 10 paces. Make a sketch of the paths where more clues may be found. Discuss the symmetry, asymptotes, increasing-decreasing, and concavity.

SOLUTIONS TO PREREQUISITE QUIZ

1. $f'(x_0) = 0$
2. $g'(x_0) > 0$
3. Increasing
4. Concave downward

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5. (a) 2

(b) 3

(c) $+\infty$

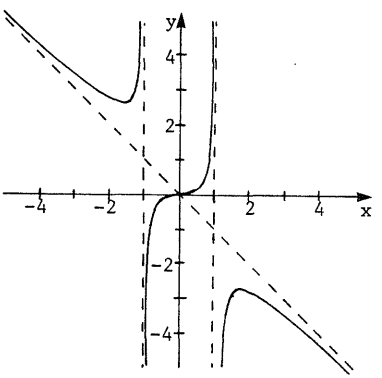
(d) $-\infty$

SOLUTIONS TO SECTION QUIZ

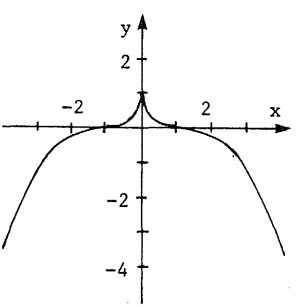
1. (a) Symmetric about origin

(b) $x = \pm 1$ (Also, $y = -x$)

(c)



2.



Critical and inflection points at $(\pm 1, 0)$; cusp at $(0,1)$.

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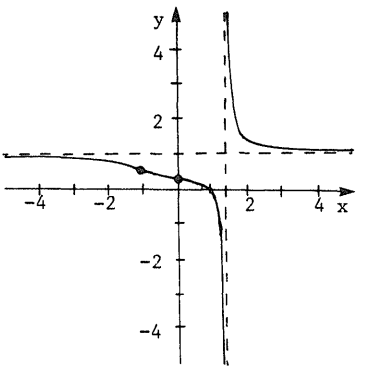
3.



There are no special symmetries; $x = \sqrt[3]{3}$ is a vertical asymptote; $y = 1$ is a horizontal asymptote; decreasing for $x \neq \sqrt[3]{3}$; concave upward on $(-\sqrt[3]{3/2},0),(\sqrt[3]{3},\infty)$; concave downward on $(-\infty,-\sqrt[3]{3/2})$, $(0,\sqrt[3]{3})$. There are no critical points; however, inflection points occur at $x = -\sqrt[3]{3/2}$ and at $x = 0$.

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3.5 Maximum-minimum Problems

PREREQUISITES

1. Recall how to find critical points (Section 3.2).
2. Recall tests used to determine maxima and minima (Section 3.2).

PREREQUISITE QUIZ

1. Classify the critical points of the following as local minima, local maxima, or neither:
   (a) $(x^2 - 3)/(x - 2)$
   (b) $(x + 3)(x^2 - x - 1)$
2. If $f(x_0)$ is a local minimum, what can you say about $f'(x_0)$? What about $f''(x_0)$?

GOALS

1. Be able to find the global minimum and the global maximum on an interval.
2. Be able to solve minimum-maximum word problems.

STUDY HINTS

1. Definitions. A local maximum is a maximum on a small interval, whereas a global maximum is a maximum on the entire interval of definition. Note that a global maximum is always a local maximum, but not vice versa. Similar comments may be made for minima.
2. Points and values. The value of a maximum refers to y and points refer to x . Whereas there can only be one value for the maximum, there may be several points where the maximum is assumed.

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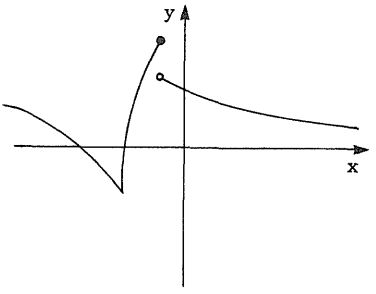
3. Extreme value theorem. Note that two conditions are necessary: (i) continuity and (ii) closed intervals. By eliminating just one condition, the guarantee of a minimum and maximum no longer exists. However, both minimum and maximum may occur even if one or none of the conditions hold. See figure.



4. Closed interval test. Essentially, it is saying, "If a function is (i) continuous and (ii) differentiable on a closed interval, then the minimum and maximum points must exist at critical points or endpoints." Think about what happens if differentiability wasn't required, as in $f(x) = |x|$. What if a discontinuity existed? What if the interval was not closed?

5. Solving word problems. The most difficult part is translating words into equations. Whenever possible, drawing a picture often helps one to visualize and understand the entire question. Write down equations to describe the situation. If you have several variables, try to rewrite them in terms of a single variable. Before finding a solution, a guess may help determine the correctness of your answer. After finding a solution, ask yourself if it makes sense. Finally, we emphasize that practice makes perfect.

6. Test taking. Calculus exams almost always have a maximum-minimum word problem. In the majority of cases, you will be asked to solve problems with a minimum of information. Thus, you should try to use all of the

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6. Test taking (continued).

numbers and equations which are given to you. You should also be aware that some instructors attempt to test your understanding by giving extraneous information.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Use a variation of the closed inteval test and analyze $\lim_{x\to\infty} f(x)$ along with the critical and endpoints. $f'(x) = 6x^2 - 5$, so the critical points are $x = \pm\sqrt{5/6}$; however, neither critical point is in $[1,\infty)$. $f(1) = -1$ and $\lim_{x\to\infty} f(x) = \infty$. Thus, 1 is the minimum point and -1 is the minimum value. There is no maximum point.

5. Use a variation of the closed interval test and analyze $\lim_{x\to-\infty} f(x)$, $\lim_{x\to 0} f(x)$, and the critical points. $f(x) = x + 1/x$, so $f'(x) = 1 - 1/x^2$, so the critical points are ± 1. $\lim_{x\to-\infty} f(x) = -\infty$, $f(-1) = -2$, and when x is near 0 and negative, the limit is $-\infty$. Thus, the maximum point is -2 and there is no minimum point in $(-\infty,0)$.

9. The minimum point is about 1971.4 and the maximum point is 1980. The minimum value is about 16%, whereas the maximum value is about 34%.

13. $f'(x) = 4x^3 - 8x$, so $x = 0,\pm\sqrt{2}$ are the critical points; -4 and 2 are the endpoints. $f(-4) = 199$; $f(-\sqrt{2}) = 3$; $f(0) = 7$; $f(\sqrt{2}) = 3$; $f(2) = 7$. Therefore, -4 is the maximum point; the maximum value is 199. $\pm\sqrt{2}$ are the minimum points and 3 is the minimum value.

17. Use a variation of the closed interval test and analyze the limits at $\pm\infty$, if necessary. $f'(x) = 2x - 3$, so the critical point is $3/2$. $\lim_{x\to-\infty} f(x) = \infty$; $f(-8) = 89$; $f(-3) = 19$; $f(-2) = 11$; $f(-3/2) = 7.75$; $f(-1) = 5$; $f(1/2) = -0.25$; $f(1) = -1$; $f(3/2) = -1.25$; $f(2) = -1$; $f(8) = 41$; $\lim_{x\to\infty} f(x) = \infty$.

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17 (continued).

(a) There are no critical points, endpoints, nor minimum and maximum points.

(b) There is no critical point. The endpoint is $1/2$, which is the minimum point. The minimum value is $-1/4$. There is no maximum point.

(c) The critical point is $3/2$ and the endpoint is 2. The minimum point is $3/2$ with minimum value $-5/4$. There is no maximum point.

(d) The answer is the same as part (c).

(e) The critical point is $3/2$. There are no endpoints. The minimum point is $3/2$ with minimum value $-5/4$. There is no maximum point.

(f) The answer is the same as part (e).

(g) There is no critical point. The endpoints are -1 and 1. The maximum point is -1 and the minimum point is 1. The maximum value is 5 and the minimum value is -1.

(h) The critical point is $3/2$. The endpoints are -8 and 8. The maximum point is -8 and the minimum point is $3/2$. The maximum value is 89 and the minimum is $-5/4$.

21. $f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$. $f'(x) = 0$ at $x = 1$, so this is the critical point. There are no endpoints. Since $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$, there are no maximum or minimum points, nor any maximum or minimum values.

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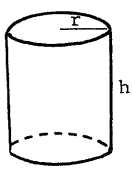
25. $f'(x) = [3x^2(x^2 + 1) - (x^3 - 1)(2x)]/(x^2 + 1)^2 = (x^4 + 3x^2 + 2x)/(x^2 + 1)^2 = x(x^3 + 3x + 2)/(x^2 + 1)^2$. Using a calculator, we find that the critical points are 0 and -0.60 . The endpoints are -10 and 10 . $f(-10) = -1001/101$; $f(-0.60) \approx -0.89$; $f(0) = -1$; $f(10) = 999/101$, so the maximum point is 10 and the maximum value is $999/101$. The minimum point is -10 and the minimum value is $-1001/101$.

29. $f'(x) = [3x^2(1 + x^2) - x^3(2x)]/(1 + x^2)^2 = (x^4 + 3x^2)/(1 + x^2)^2$, which equals 0 at $x = 0$, so this is the critical point. There are no endpoints. $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$, so there are no maximum or minimum points.

33. Let the two individual masses be x and y . We are given that $x + y = M$, and that the force is proportional to xy . We want to maximize xy , i.e., maximize $f(x) = x(M - x)$.

37. $f(x) = x(M - x)$, so $f'(x) = M - 2x = 0$ if $x = M/2$. Also, $f''(M/2) = -2 < 0$, which implies that $x = M/2$ is a maximum point. Hence, the two masses should be equal.

41. (a) 

We are given $V = \pi r^2 h = 1000$ and we want to minimize the surface area, $A = 2(\pi r^2) + 2\pi rh$. $\pi r^2 h = 1000$ implies $h = 1000/\pi r^2$, so $A = 2\pi r^2 + 2000/r$. Thus, $A'(r) = 4\pi r - 2000/r^2 = 0$ implies $4\pi r^3 = 2000$, so $r = \sqrt[3]{500/\pi}$ cm. and $h = 1000/\pi(500/\pi)^{2/3} = 2\sqrt[3]{500/\pi}$ cm.

(b) $V = \pi r^2 h$ implies $h = V/\pi r^2$, so $A = 2\pi r^2 + 2V/r$. Therefore, $A'(r) = 4\pi r - 2V/r^2 = 0$ implies $4\pi r^3 = 2V$, so $r = \sqrt[3]{V/2\pi}$ cm. and $h = V/\pi(V/2\pi)^{2/3} = \sqrt[3]{4V/\pi} = 2\sqrt[3]{V/2\pi}$ cm.

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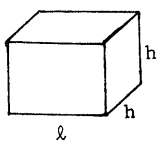
41 (continued).

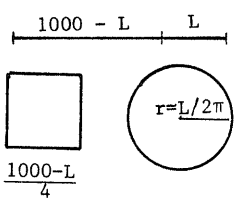
(c) Here, A is constant and we want to maximize V. $A = 2\pi r^2 + 2\pi rh$ implies $h = (A - 2\pi r^2)/2\pi r$, so $V = (A - 2\pi r^2)r/2 = (Ar - 2\pi r^3)/2$ and $V'(r) = (A - 6\pi r^2)/2$. Thus, $V'(r) = 0$ when $r = \sqrt{A/6\pi}$ cm. and $h = (2A/3)/(2\pi\sqrt{A/6\pi}) = A\sqrt{6\pi}/3\pi\sqrt{A} = \sqrt{2A/3\pi}$ cm.

45.



We are given that $\ell + 2h + 2h = \ell + 4h = 72$, and we want to maximize $V = \ell h^2$. $\ell + 4h = 72$ implies $\ell = 72 - 4h$, so $V = (72 - 4h)h^2 = 72h^2 - 4h^3$, and $V'(h) = 144h - 12h^2 = 12h(12 - h)$. Thus, $V'(h) = 0$ when $h = 0$ or $h = 12$, but $h = 0$ means $V = 0$, so h must be 12. Thus, the height and width are 12 inches, while the length is 24 inches.

49.



Use $(1000 - L)$ feet of fencing for the square area and L feet of fencing for the circular area. Each side of the square is $250 - L/4$ and the circle has a radius of $L/2\pi$. Therefore, the total area, $A = (250 - L/4)^2 + \pi(L/2\pi)^2$, so $A'(L) = 2(250 - L/4)(-1/4) + \pi(2)(L/2\pi)(1/2\pi) = -125 + L/8 + L/2\pi = (-1000\pi + L\pi + 4L)/8\pi$. $A'(L) = 0$ when $L = 1000\pi/(\pi + 4)$. $A''(L) = 1/8 + 1/2\pi > 0$, so the minimum area is attained at the critical point.

(a) The only critical point is a minimum, so the maximum occurs at the endpoints. When all of the fencing is used for the square area, the total area is $(250)^2 = 62{,}500$ square feet. If all of the fencing is used for the circular area, the total area is

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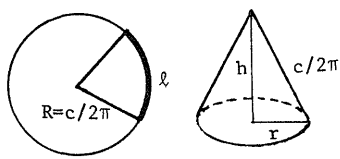
49 (continued).

(a)(continued) $\pi(500/\pi)^2 = 250{,}000/\pi \approx 79{,}577.5$ square feet, so the maximum occurs when all of the fencing is used for the circular area, which has a radius of $(500/\pi)$ feet.

(b) The minimum area is attained at the critical point. The side of the square is $[250 - 250\pi/(\pi + 4)] = [1000/(\pi + 4)]$ feet and the radius of the circle is $[500/(\pi + 4)]$ feet. The total area is $[1000/(\pi + 4)]^2 + \pi[500/(\pi + 4)]^2 = [4(500)^2 + \pi(500)^2]/(\pi + 4)^2 = [250{,}000/(\pi + 4)]$ square feet.

53. 

From geometry, we know that $c = 2\pi R$, so $R = c/2\pi$, which is the slant height of the dunce cap. The circumference of the cap's base is $c - \ell$, so $r = (c - \ell)/2\pi$, and we also know that $h^2 = (c/2\pi)^2 - r^2$. We want to maximize $V = \pi r^2 h/3$, but it is easier to maximize $V^2 = \pi^2 r^4 h^2/9 = \pi^2[(c - \ell)/2\pi]^4[(c/2\pi)^2 - ((c - \ell)/2\pi)^2]/9 = (c - \ell)^4(2c\ell - \ell^2)/576\pi^4$. Thus, $(V^2)'(\ell) = [4(c - \ell)^3(-1)(2c\ell - \ell^2) + (c - \ell)^4(2c - 2\ell)]/576\pi^4 = 2(c - \ell)^3(-4c\ell + 2\ell^2 + c^2 - 2c\ell + \ell^2)/576\pi^4 = 2(c - \ell)^3(3\ell^2 - 6c\ell + c^2)/576\pi^4$. To maximize V^2, we solve $0 = 3\ell^2 - 6c\ell + c^2$, which has the solutions $\ell = (6c \pm 2\sqrt{6}c)/6$. $(3 + \sqrt{6})c/3$ gives us $r < 0$, so the answer is $\ell = (3 - \sqrt{6})c/3$.

57. This problem requires the maximizations of area. If we reflect the shape of the fencing across the shoreline, we have a symmetric figure. A circle, the most symmetric of all geometric figures, should maximize area if the perimeter is held constant. Therefore, the 500 feet of

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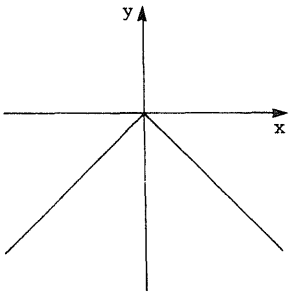
57 (continued).

fencing should be used to make a semicircle with a radius of $(500/\pi)$ meters. The maximum area is $(125{,}000/\pi)$ square meters.

61. 

One possibility is a function which comes to a point at its maximum. A simple example is $y = -|x|$. There is no unique solution.

65. Suppose M_1 and M_2 are both maximum values of f on I . So there exists at least one x_1 in I such that $f(x_1) = M_1$. M_2 being a maximum value implies $M_2 \geqslant f(x)$ for all x in I , in particular $M_2 \geqslant f(x_1) = M_1$ (Eq.1) .

On the other hand, M_2 being a maximum value of f on I implies that there exists an x_2 in I such that $f(x_2) = M_2$. M_1 being a maximum value implies $M_1 \geqslant f(x_2) = M_2$ (Eq.2) . Combining (Eq. 1) and (Eq. 2) we have $M_1 = M_2$, i.e., the maximum value of f in I is unique.

69. $f'(x) = [(1 - x^2) - x(-2x)]/(1 - x^2)^2 = (1 + x^2)/(1 - x^2)^2$ for $-1 < x < 0$ and $f'(x) = 3x^2 - 1$ for $0 < x < 2$. $f'(0)$ does not exist. The critical point is $\sqrt{1/3}$ and there are no endpoints. For x near -1 and $x > -1$, $\lim_{x \to -1} f(x) = -\infty$; $f(0) = 0$; $f(\sqrt{1/3}) = -2/3\sqrt{3} \approx -0.38$; $\lim_{x \to 2} f(x) = 6$. Since $f(x)$ only approaches 6 , $f(x)$ has no maximum nor minimum values.

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73. (a) At the supermarket, the cheapest price is 32.5 cents per quart if one buys a half gallon. At the corner grocery, a half gallon of milk costs 80 cents; however, one can buy a quart today and then a half gallon tomorrow which costs 77.5 cents for the first two quarts, so a quart should be purchased today.

(b) As shown in part (a), a half gallon should be bought if $g/2 \leqslant q$ and $G \leqslant Q + g/2$. If $q \leqslant g/2$, as when a sale occurs, then a half gallon should be bought if $G \leqslant Q + q$.

77. By the definition of concavity (Section 3.3), f is differentiable on (a,b) and f lies above its tangent lines, locally. Thus, for x near x_0, we have $f(x) > f(x_0) + f'(x_0)(x - x_0)$. Since f is differentiable, the closed interval test tells us that the maximum must be a critical point or an endpoint. If x_0 is a critical point, then $f(x) > f(x_0)$, so x_0 is a minimum. Therefore, the maximum point of f is an endpoint.

SECTION QUIZ

1. This question demonstrates the limitations of the extreme value theorem.

(a) Sketch the graph of a function on a closed interval which lacks a minimum and a maximum.

(b) Sketch the graph of a continuous function which has neither a minimum nor a maximum.

(c) Sketch the graph of a discontinuous function on an open interval which contains both a minimum and a maximum.

2. True or false:

(a) A global maximum must be a local maximum.

(b) If a function contains several local minima, the smallest of these is the global minima.

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3. Crazy Charlie challenged a city bus to a demolition derby. Crazy Charlie's little compact will have a value of $c(t) = 12000 - 30t^2$, t minutes into the derby. The value of the bus will be $b(t) = 20000 - 10t^2 - 80t$. Crazy Charlie quits when his car is totaled and has a value of zero. Find the minimum and maximum differences between the values of the vehicles, $|c(t) - b(t)|$.

SOLUTIONS TO PREREQUISITE QUIZ

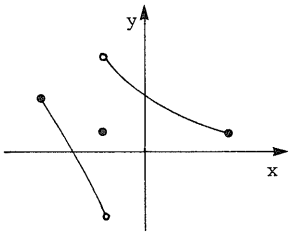
1. (a) local maximum: 1 ; local minimum: 3

   (b) local maximum: -2 ; local minimum: 2/3

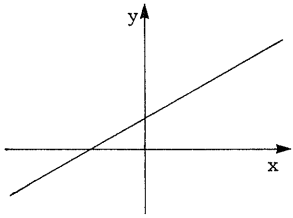
2. $f'(x_0) = 0$; $f''(x_0) \geqslant 0$.

SOLUTIONS TO SECTION QUIZ

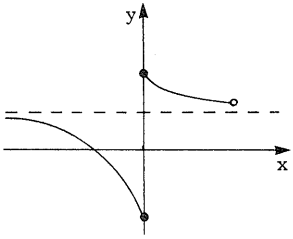
1. (a)



(b)



(c)



2. True.

3. Maximum = \$14400 ; minimum = \$7920 .

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3.6 The Mean Value Theorem

PREREQUISITES

1. Recall how to find an average rate of change (Section 1.1).
2. Recall the concept of continuity (Sections 1.2 and 3.1).
3. Recall the critical point test (Section 3.2).

PREREQUISITE QUIZ

1. A tour bus travels $4t^{3/2}$ kilometers in t hours. What is the average velocity during the interval $1 \leqslant t \leqslant 4$?
2. If $f(x_0)$ exists, what other condition is necessary for continuity?
3. If $f(x_0)$ is a local minimum or a local maximum, what can you say about $f'(x_0)$, assuming that $f'(x)$ exists?

GOALS

1. Be able to understand the mean value theorem.
2. Be able to estimate the difference between function values, given information about the derivative.

STUDY HINTS

1. Mean value theorem. The theorem states that the average slope is attained somewhere in an interval, possibly at more than one point, if two conditions hold: i) the function is continuous on $[a,b]$ and ii) it is differentiable on (a,b). Example 3 explains what happens if differentiability doesn't hold.

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2. Typical application. Example 2 shows a typical proof which relies on getting an expression of the form $[f(x_1) - f(x_0)]/(x_1 - x_0)$. Many times, as in Example 2, you will simply be asked to multiply through by the denominator to estimate the difference between function values.

3. Rolle's theorem. Basically, it asserts that if $f = 0$ at two points, then the derivative is zero somewhere in between, assuming continuity and differentiability.

4. Horserace theorem. This says that if two continuous, differentiable functions have equal values at the endpoints, then they must have equal slopes somewhere in between.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Since f is continous on $[0,1/2]$ and $f'(x)$ exists for x in $(0,1/2)$, the mean value theorem applies. It states that there exists x_0 in $(0,1/2)$ such that $f'(x_0) = [f(1/2) - f(0)]/(1/2 - 0)$. Since $0.3 \leq f'(x) \leq 1$ for x in $(0,1/2)$, we have, in particular, $0.3 \leq f'(x_0) \leq 1$, i.e., $0.3 \leq [f(1/2) - f(0)]/(1/2) \leq 1$ or $0.15 \leq f(1/2) - f(0) \leq 1/2$.

5. Since $f(x)$ is a polynomial, it is continous and differentiable. By the mean value theorem, there is an x_0 in $(-1,0)$ such that $f'(x_0) = [f(0) - f(-1)]/(0 - (-1)) = (15 - 17)/1 = -2$.

9. Let $x(t)$ be the position of the object. We are given that the velocity of the object $x'(t)$ is such that $34 \leq x'(t) \leq 36$. Antidifferentiation gives $34t + C \leq x(t) \leq 36t + C$. $x(0) = 4$ gives $C = 4$. Hence, $34t + 4 \leq x(t) \leq 36t + 4$ and so $72 \leq x(2) \leq 76$.

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13. The derivative of $1/x$ is $-(1/x^2)$, so by Consequence 3, $F(x) = (1/x) + C$, where C is a constant, on any interval not containing zero. So $F(x) = (1/x) + C_1$ for $x < 0$ and $F(x) = (1/x) + C_2$ for $x > 0$. Hence C_1 and C_2 are constants, but they are not necessarily equal.

17. By the power and polynomial rules for antidifferentiation, we get $(1/2)x^2/2 - 4x^3/3 + 21x + C = x^2/4 - 4x^3/3 + 21x + C$.

21. By the power rule for antidifferentiation, we get $F(x) = 2x^5/5 + C$, but $F(1) = 2/5 + C = 2$ implies $C = 8/5$. Thus, $F(x) = 2x^5/5 + 8/5$.

25. Let c, d, and e be the points with $a < c < d < e < b$ in (a,b) at which f vanishes, then $[f(d) - f(c)]/(d - c) = 0$ and $[f(e) - f(d)]/(e - d) = 0$ implies that the first derivative vanishes somewhere in (c,d) and somewhere in (d,e) according to the mean value theorem. Now, let m and n be the points in (a,b) where $f'(x)$ vanishes. By the mean value theorem, there exists an x_0 in (m,n) such that $f''(x_0) = [f'(n) - f'(m)]/(n - m) = 0$. Since (m,n) is in (a,b), the statement is proven.

29. By applying the mean value theorem to $[t_1,t_2]$, we have $[N(t_2) - N(t_1)]/(t_2 - t_1) = 0$ because $N(t_2) = N(t_1)$. Therefore, dN/dt cannot be greater than 0 on all of (t_1,t_2). Also, N is not constant, so dN/dt cannot be 0 on the entire interval. Consequently, $dN/dt < 0$ for some T in (t_1,t_2), which means that the population is decreasing. Do the same for the interval $[t_2,t_3]$.

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SECTION QUIZ

1. (a) What are the two important conditions which are required before the mean value theorem can be applied?

   (b) What equation describes the statement of the mean value thoerem?

   (c) Suppose $f(x) = (x^3 - 3)/(x + 2)$. Show that somewhere in $(0,2)$, the slope of $f(x)$ is $11/8$.

2. Consider the function $g(t) = 1/t^2$ on $(-1,3)$. Then $[g(3) - g(-1)]/(3 - (-1)) = -2/9$. Now, $g'(t) = -2/t^3$, which is $-2/9$ at $t_0 = \sqrt[3]{9}$. How can the mean value theorem be satisfied when there is a discontinuity at $t = 0$?

3. Mean Michael's favorite pastime was playing with little girl's ponytails. When he was in the third grade, Michael would sneak up behind his schoolmates during recess, yank their ponytails, and tie their hair to a low-hanging tree branch. Michael's technique imporved as the week went by and during the weekend, he became well-behaved with his parents. After t days of school during the week, Mean Michael plays with approximately t^2 girl's ponytails.

   (a) Use the mean value theorem to show that at some time during the week, he yanks ponytails at a rate of about 5 per day.

   (b) Approximately when is the rate about 5 per day?

ANSWERS TO PREREQUISITE QUIZ

1. 28/3 km/hr .

2. $\lim_{x \to x_0} f(x) = f(x_0)$.

3. $f'(x_0) = 0$

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ANSWERS TO SECTION QUIZ

1. (a) Continuity and differentiability .

   (b) $f'(x_0) = [f(b) - f(a)]/(b - a)$.

   (c) Use the mean value theorem. $f(x)$ is continuous and differentiable on $(0,2)$; $[f(2) - f(0)]/2 = 11/8$.

2. This is a case in which the conclusion is true even though the hypotheses are not met. Meeting the hypotheses guarantees the conclusion, but a conclusion may be true even without meeting the hypotheses.

3. (a) $[f(5) - f(0)]/(5 - 0) = 5$.

   (b) At about $5/2$ days into the week.

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3.R Review Exercises for Chapter 3

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The function is continuous whenever the denominator exists and does not equal zero. We need $x^2 - 1 > 0$, so $f(x)$ is continuous in $(-\infty,-1)$ and $(1,\infty)$.

5. $h(2) = -(2)^3 + 5 = -3$. From the left, $\lim_{x\to 2-} h(x) = -(2)^3 + 5 = -3$. From the right, $\lim_{x\to 2+} h(x) = (2)^2 - 7 = -3$. The limit of h at $x = 2$ exists and is equal to the value of h at $x = 2$, so h is continuous at $x = 2$. h is continuous on the whole real line because the function is continuous at $x = 2$ and all polynomials are continuous.

9. $f(0) = -2$ and $f(1) = 1$. By the intermediate value theorem (version 1), $f(x)$ has a root in $(0,1)$.

13. $f'(x) = 24x^2 - 6x = 6x(4x - 1) = 0$ at $x = 0,1/4$. $f'(x) > 0$ on $(-\infty,0)$ and $(1/4,\infty)$, so it is increasing on these intervals. $f'(x) < 0$ on $(0,1/4)$, so it is decreasing on this interval.

17. $B(t) = t^3/10 - t^2/2 - 5t/2 + 10$, $-3 \leq t \leq 7$ (in this particular problem, a negative value of t is possible for it is used to represent a time before the reference time, January 1, 1993). $B'(t) = 3t^2/10 - t - 5/2 = 0$ at $t = -5/3$ and 5 in $[-3,7]$. $B'(t) < 0$ on $(-5/3,5)$; $B'(t) > 0$ on $[-3,-5/3)$ and $(5,7]$. So B is decreasing on $(-5/3,5)$, increasing on $[-3,-5/3)$ and $(5,7]$. $B''(t) = 3t/5 - 1 = 0$ at $t = 5/3$; $B'''(5/3) = 3/5 \neq 0$, so $5/3$ is an inflection point. $B'''(t) > 0$ for t in $(5/3,7]$ and $B''(t) < 0$ for t in $[-3,5/3)$. So 5 is a local minimum and $-5/3$ is a local maximum. 1990 will begin with a high inflation rate of 10.3% that will keep on increasing until it hits a record high of 15.1% (maximum) approximately one and one-third years later when luck begins to turn, i.e., the inflation rate will begin to decrease. To the joys of the politicians, this trend will continue steadily,

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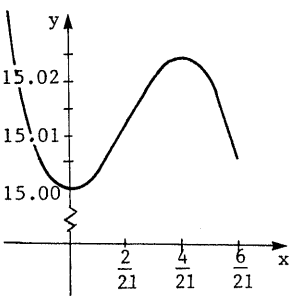
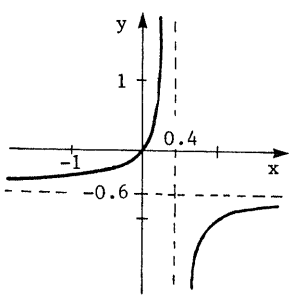
17 (continued).

with even an accelerated decrease in September of 1994 (inflection point), and reaching their goal of 0% inflation rate in 1996. So they will be quite surprised to see a change of events around 1995 when the inflation rate begins to increase for the first time since mid-1991. This increase continues to the year 2000.

21. Since $f'(x) = 6x^2 - 10x + 4 = 2(3x - 2)(x - 1)$, the critical points are $2/3$ and 1. $f''(x) = 12x - 10$. Since $f''(2/3) = -2 < 0$, $2/3$ is a local maximum. And since $f''(1) = 2 > 0$, 1 is a local minimum.

25. 

$f(x)$ is a polynomial, so it is continuous on $(-\infty,\infty)$, and it is differentiable on $(-\infty,\infty)$. $f'(x) = -21x^2 + 4x = x(-21x + 4)$, so the critical points are $x = 0, 4/21$. $f''(x) = -42x + 4 = 0$ at $x = 2/21$. $f'''(x) = -42$. Thus f is increasing on $(0,4/21)$ and decreasing on $(-\infty,0)$ and $(4/21,\infty)$. It is concave upward on $(-\infty,2/21)$ and concave downward on $(2/21,\infty)$. There are no endpoints. There is a local maximum at $4/21$ and a local minimum at 0. $2/21$ is an inflection point.

29. 

The domain is $x \neq 2/5$. $f(x)$ is continuous and differentiable on $(-\infty,2/5)$ and $(2/5,\infty)$. $f'(x) = [3(2 - 5x) - 3x(-5)]/(2 - 5x)^2 = 6/(2 - 5x)^2$, so there are no critical points and $f(x)$ is increasing throughout its domain. $f''(x) = -12(2 - 5x)^{-3}(-5) = 60/(2 - 5x)^3$, so there is no inflection point. f is concave upward on $(-\infty,2/5)$ and concave downward

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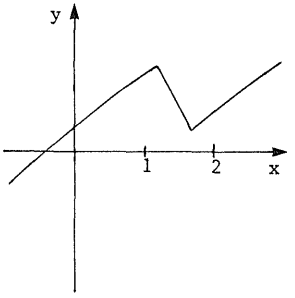
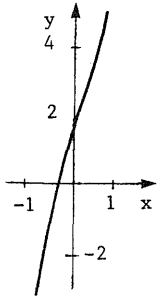
29 (continued).

on $(2/5,\infty)$. There are no endpoints, local maxima, nor local minima. Note that $x = 2/5$ is a vertical asymptote and $y = 3/5$ is a horizontal asymptote.

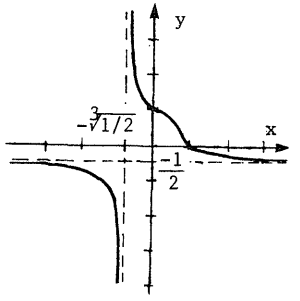
33. 

The statement is false. The figure shows that the function may be decreasing within the interval $(1,2)$.

37. This is true. Let $f(x) = ax^2 + bx + c$, $a \neq 0$. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$, which never vanishes. Thus, parabolas don't have inflection points.

41. 

$f'(x) = 3x^2 + 3$ is positive for all x , so f is increasing on $(-\infty,\infty)$. $f''(x) = 6x$ and $f'''(x) = 6$, so $x = 0$ is an inflection point. $f(x)$ is concave downward on $(-\infty,0)$ and concave upward on $(0,\infty)$.

45. 

There is a vertical asymptote at $x = -\sqrt[3]{1/2}$. $\lim_{x\to\pm\infty} f(x) = -1/2$ is a horizontal asymptote. $f'(x) = [(-3x^2)(1 + 2x^3) - (1 - x^3)(6x^2)]/(1 + 2x^3)^2 = -9x^2/(1 + 2x^3)^2$, so $x = 0$ is a critical point. $f''(x) = [-18x(1 + 2x^3)^2 - (-9x^2)(2)(1 + 2x^3)(6x^2)]/(1 + 2x^3)^4 =$

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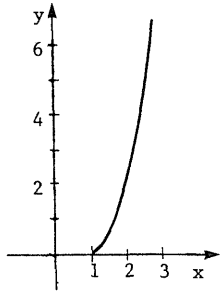
45 (continued).

$18x(4x^3 - 1)/(1 + 2x^3)^3$, so possible inflection points are 0 and $\sqrt[3]{1/4}$. f'' changes sign at 0 and $\sqrt[3]{1/4}$, so these are inflection points. f is decreasing on its domain. It is concave upward on $(-\sqrt[3]{1/2},0)$ and $(\sqrt[3]{1/4},\infty)$; concave downward on $(-\infty,-\sqrt[3]{1/2})$ and $(0,\sqrt[3]{1/4})$.

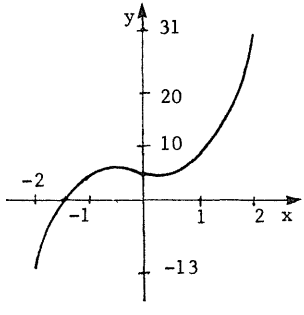
49.



$y = x\sqrt{(x-1)^3}$ implies $(x-1)^3 \geq 0$, so the domain is $x \geq 1$. $f'(x) = (x-1)^{3/2} + x(3/2) \times (x-1)^{1/2} = (5x/2 - 1)(x-1)^{1/2}$ implies the graph is strictly increasing. $f''(x) = 5(x-1)^{1/2}/2 + (5x/2 - 1)(1/2)(x-1)^{-1/2} = (15x - 2)/4\sqrt{x-1}$ implies the graph is concave up.

53. Use the closed interval test of Section 3.5. $f'(x) = [(20x^3)/(x^2+1) - (5x^4 + 1)(2x)]/(x^2+1)^2 = 2x(5x^4 + 10x^2 - 1)/(x^2+1)^2$. Using the quadratic formula, we find the critical points to be $\pm[(-5 + \sqrt{30})/5]^{1/2}$ and 0 . The endpoints are -1 and 1 . $f(\pm 1) = 3$; $f(0) = 1$; $f[\pm((-5 + \sqrt{30})/5)^{1/2}] = 2\sqrt{30} - 10 \approx 0.95$, so the maximum value is 3 and the minimum value is $2\sqrt{30} - 10$.

57.



This function has no symmetry. $f'(x) = 9x^2 + 2x - 1$, which is zero at $x = (-2 \pm \sqrt{40})/18 = (-1 \pm \sqrt{10})/9$. $f''(x) = 18x + 2$, so the inflection point is at $x = -1/9$. At $x = (-1 - \sqrt{10})/9 \approx -0.46$, $f''(x) < 0$, so this is a local maximum. At $x = (-1 + \sqrt{10})/9 \approx 0.24$, $f''(x) > 0$, so this is a local minimum.

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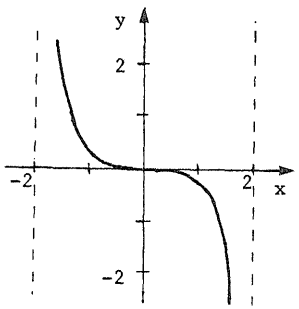
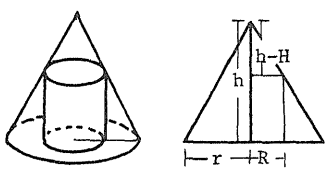
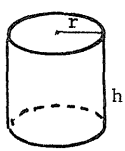
61. 

The graph is symmetric to the origin. Vertical asymptotes exist at $x = \pm 2$. $f'(x) = [(3x^2)(x^2 - 4) - x^3(2x)]/(x^2 - 4)^2 = x^2(x^2 - 12)/(x^2 - 4)^2$. The only critical point in $(-2,2)$ is $x = 0$, but $f'(x) \geqslant 0$ for all x in $(-2,2)$. $f''(x) = [(4x^3 - 24x)(x^2 - 4)^2 - (x^4 - 12x^2)(2) \times (x^2 - 4)(2x)]/(x^2 - 4)^4 = 4x(2x^2 + 24)/(x^2 - 4)^3$, so the only possible inflection point is $x = 0$.

65. 

Let r and h be the radius and height of the cone, and let R and H be the radius and height of the cylinder. We want to maximize $V = \pi R^2 H$. By similar triangles, we have $R/(h - H) = r/h$, so $R = r(h - H)/h$, and $V = \pi r^2(h - H)^2 H/h^2$. Thus, $V'(H) = (\pi r^2/h^2)[2(h - H)(-1)H + (h - H)^2]$, and $V'(H) = 0$ implies $2H(h - H) = (h - H)^2$ or $2H = h - H$. Therefore, $H = h/3$ and $R = r(h - h/3)/h = 2r/3$.

69. 

We are given that $V = \pi r^2 h$ and we want to minimize the cost, $C = (1/20)(2\pi rh + 2\pi r^2) + p(2)(2\pi r)$. $V = \pi r^2 h$ implies $h = V/\pi r^2$, so $C = (1/20)(2V/r + 2\pi r^2) + 4p\pi r = V/10r + \pi r^2/10 + 4p\pi r$. $C'(r) = -V/10r^2 + \pi r/5 + 4p\pi = (-V + 2\pi r^3 + 40p\pi r^2)/10r^2$. Therefore, the cheapest can has a radius given by $V = 2\pi r^3 + 40p\pi r^2$ and the height is given by $h = V/\pi r^2$, where the dimensions are given in centimeters.

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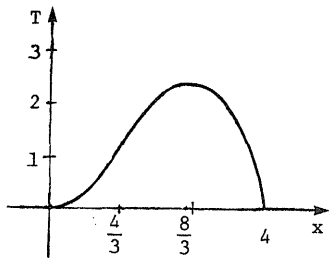
73. (a) $T(x) = x^2(1 - x/4)$, $T'(x) = x(2 - 3x/4)$, $T'(2) = 1$.

(b) 

$T'(x) = 0$ at $x = 0, 8/3$. $T''(x) = 2 - 3x/2 > 0$ for $x < 4/3$ and $T''(x) < 0$ for $x > 4/3$, so T is concave upward on $[0, 4/3)$ and concave downward on $(4/3, 4]$. Also $T''(0) > 0$ and $T''(8/3) < 0$, so $x = 0$ is a local minimum and $x = 8/3$ is a local maximum. $T''(4/3) = 0$ and $T'''(4/3) \neq 0$, so $4/3$ is an inflection point. $T(0) = T(4) = 0$; $T(8/3) = 64/27$; $T(4/3) = 32/27$.

77. (a) Revenue = price × quantity; total cost = fixed cost + cost of production where cost of production is cost per unit times number of units made. Profit = money taken in less the money used for production. Then substitute for R and C .

(b) $0 = (4.5)x - 9000$ has the solution $x = 2000$ units.

(c) $4500 = (4.5)x - 9000$ has the solution $x = 3000$ units.

(d) $P = R - C = [16x - (4/9)10^{-6}x^2] - [9000 + 11.5x] = -(4/9)10^{-6}x^2 + 4.5x - 9000$, so $P'(x) = -(8/9)10^{-6}x + 4.5$ and $P''(x) = -(8/9)10^{-6}$. $P'(x) = 0$ at $x = (81/16)10^6 = 5{,}625{,}000$. This maximizes profits since $P''(x) < 0$.

81. (a) $C'(x) = -2kx^{-3} + (-2)(3k)(40 - x)^{-3}(-1) = -2k(x^{-3} - 3(40 - x)^{-3}) = -2k[(40 - x)^3 - 3x^3]/x^3(40 - x)^3$. The critical points are found by solving $(40 - x)^3 = 3x^3$, which implies $(40 - x) = \sqrt[3]{3}x$ or $x = 40/(\sqrt[3]{3} + 1) \approx 16.4$. There are no other critical points.

(b) The critical point is the distance in miles from Smellter where particulate matter is minimized; therefore, the house should be built 16.4 miles from Smellter.

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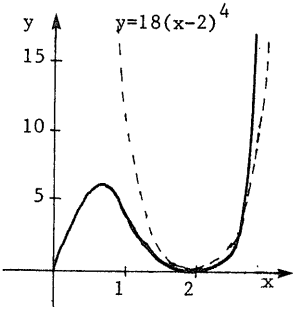
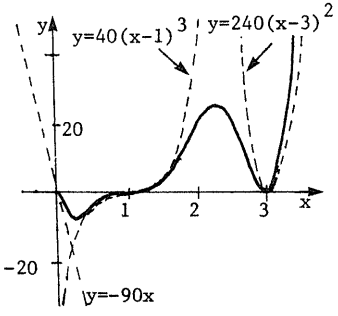
85. (a) 5 feet, 2 inches = 62 inches, so $W = (1/2)(62/10)^3 = 119.164$ pounds.

(b) $W'(h) = (3/2)(h/10)^2(1/10) = 3h^2/2000$, so $W'(48) = 432/125$ and $W'(50) = 15/4$. By Consequence 1 of the mean value theorem, we have $432/125 < [W(b) - W(a)]/(b - a) < 15/4$. Since $b - a = 2$, $864/125 = 6.912 < W(50) - W(48) < 15/2 = 7.5$; therefore, the child gains between 6.912 and 7.5 pounds. By direct calculation, $W(48) = 55.296$ and $W(50) = 62.5$, so $W(50) - W(48) = 7.204$ pounds.

89. (a)



(b)



$y = 10x(x - 1)^3(x - 3)^2$. For x near 0, $(x - 1)^3(x - 3)^2 \approx (-1)^3(-3)^2 = -9$, so $y \approx -90x$. For x near 1, $10x(x - 3)^2 \approx 10(-2)^2 = 40$, so $y \approx 40(x - 1)^3$. For x near 3, $10x(x - 1)^3 \approx 30(2)^3 = 240$, so $y \approx 240(x - 3)^2$. The graphs of these three functions give the shape of the graph of the factored polynomial near its three roots (see the dotted curves). Use the usual graphing procedure to complete the graph of y on $[0,4]$.

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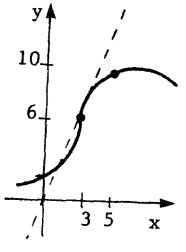
93. As f is a polynomial of degree $n > 0$, f' is a polynomial of degree $n - 1 \geqslant 0$ with a finite number of roots in $(0,1)$ (at most $n - 1$), say $x_1 < x_2 < \ldots < x_{n-1}$. (Note: In the case of $n - 1 = 0$, $f' =$ constant $\neq 0$ since f is nonconstant, which eliminated the only case where f' may have infinite number of roots.) As f' is continuous on $(0,1)$, f is strictly increasing or decreasing on each of the intervals between successive critical points. Suppose f has no local maxima or minima in $(0,1)$, then f' does not change sign in $(0,1)$. So f is increasing or decreasing on each of the intervals $(0,x_1)$, (x_1,x_2), ..., $(x_{n-1},1)$. By the continuity of f, f is increasing (decreasing) on each of the closed intervals $[0,x_1]$, $[x_1,x_2]$, ..., $[x_{n-1},1]$ and hence, on $[0,1]$. So $f(0) < f(1)$ or $f(0) > f(1)$, which contradicts the hypothesis $f(0) = f(1)$.

97. 

Let $g(x) = [f(x)]/x$, then the mean value theorem applied to $g(x)$ on $[3,5]$ gives us $(10/5 - 6/3)/(5 - 3) = 0$, so $g'(x_0) = 0$ for some x_0 in $(3,5)$. We also know that $g'(x_0) = [x_0 f'(x_0) - f(x_0)]/x_0^2$, so $x_0 f'(x_0) = f(x_0) = y_0$ for some x_0 in $(3,5)$. This satisfies the general equation of a line passing through the origin, which is $y = xf'(x)$.

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TEST FOR CHAPTER 3

1. True or false.

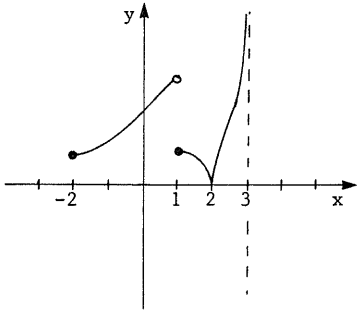
   (a) The intermediate value theorem tells you that $f(x) = 1/x$ has a root in $(-10,5)$ since $f(-10) < 0$ and $f(5) > 0$.

   (b) Any function that has a global minimum must have a local minimum.

   (c) A continuous function on a closed interval must have a local maximum.

   (d) If $f(x) = |3x|$, $f'(-1)$ does not exist.

   (e) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ is a local minimum.

2. 

   (a) Is the figure shown at the left the graph of a function?

   (b) Where is the graph continuous?

   (c) Where is the function differentiable?

3. Give a graphical example to disprove this statement. "If $f(a) = \ell$, then $\lim_{x \to a} f(x)$ is always equal to ℓ ." Under what conditions is the statement true?

4. Show that the slope of the graph of $5x^3 + 9x^2 + 15x - 3$ must be 77 somewhere in $(1,2)$ by two different methods.

   (a) Use the mean value theorem.

   (b) Use the intermediate value theorem.

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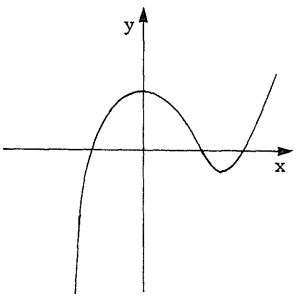
5. Use the six-step method to sketch the graph of $y = x^3/(x^2 + 1)$.

6. Use the six-step method to sketch the graph of $y = (1 - x^2)/(x + 2)$.

7. 

The sketch at the left depicts $f'(x)$. If $f(0) = 2$, make a sketch of the graph of $f(x)$.

8. Find the maximum and minimum, if they exist, of $f(x) = 3x^3 - 3x^2 + 2$ on the following intervals:

   (a) $[0,10]$

   (b) $[-3,10)$

   (c) $(-1,1)$

9. Sketch the graph of $x^{2/3} + x^2$.

10. You're running out of money and you need to get back home. It's much cheaper to mail yourself than to fly home; however, you need to be shipped in a crate. The crate must be 6 feet high. A friend will give you 100 square feet of wood for the sides and bottom. Another material is used for the top. Maximize the volume of the crate. (NOTE: People have tried this stunt in the past and suffocated.)

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ANSWERS TO CHAPTER TEST

1. (a) False; there is a discontinuity in $(-10,5)$.

   (b) True

   (c) True

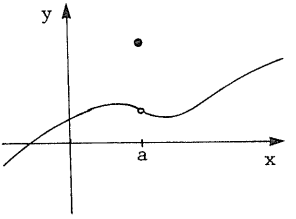
   (d) False; $f'(-1) = -3$

   (e) True

2. (a) Yes

   (b) $(-2,1)$ and $(1,3)$

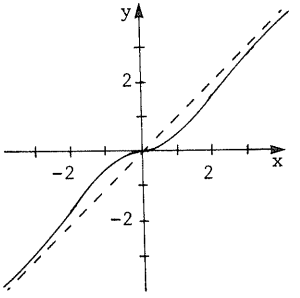
   (c) $(-2,1)$, $(1,2)$, and $(2,3)$

3. 

   The statement is true for all continuous functions.

4. (a) $[f(2) - f(1)]/(2 - 1) = (103 - 26)/1 = 77.$

   (b) $f' = 15x^2 + 18x + 15$ is continuous; $f'(1) = 48 < 77 < 93 = f'(2)$.

5. 

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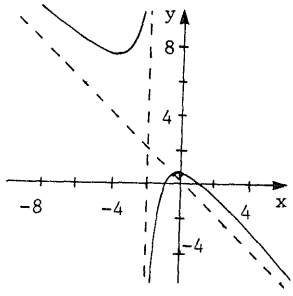
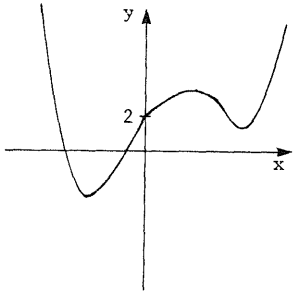
6.



7.

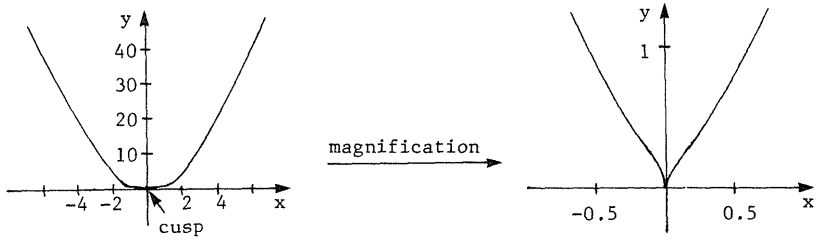


8. (a) Maximum = 2702 ; minimum = 14/9 .

(b) No maximum; minimum = -106 .

(c) Maximum = 2 ; no minimum.

9.



10. Dimensions = $(-12 + 2\sqrt{61}) \times (-12 + 2\sqrt{61}) \times 6$;

volume = $(2328 - 288\sqrt{61})$ cubic feet.

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COMPREHENSIVE TEST FOR CHAPTERS 1-3 (Time limit: 3 hours)

1. True or false. If false, explain why.

   (a) The slope of $f(x)$ at $(1,f(1))$ is $f'(1)$.

   (b) $(d/dx)[(3x^2 + x)(2x^4 + x)] = (6x + 1)(8x^3 + 1)$.

   (c) Suppose that $f(x) = \pi^3 + x^4 - 2x^2 + 1$, then $f'(x) = 3\pi^2 + 4x^3 - 4x$.

   (d) If $(a,f(a))$ is the maximum of a differentiable function on $(-\infty,\infty)$, then $f'(a)$ can only be zero.

   (e) Suppose that. $u = h(z) = z^3 - 4z$, then the tangent line at $z = 2$ is $u = (3z^2 - 4)(z - 2)$.

   (f) The most general antiderivative of $(5x^2 + 3)(25x^2 + 3)$ is $x(5x^2 + 3)^2 + C$.

   (g) A local minimum of $f(x)$ may have a value larger than a local maximum of $f(x)$.

   (h) All continuous functions are differentiable.

   (i) The product of two increasing functions is also an increasing function.

2. This question refers to $f(x) = (x^2 - x + 3)/(x + 2)$.

   (a) Is $f(x)$ symmetric to the x-axis, y-axis, origin, or neither?

   (b) Where are the asymptotes of $f(x)$?

   (c) Where does $f(x) = 0$?

   (d) On what intervals is $f(x)$ increasing? decreasing?

   (e) On what intervals is $f(x)$ concave upward? concave downward?

   (f) Where are the inflection points?

   (g) Sketch the graph of $f(x)$.

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3. Short answer.

(a) Express $f'(x)$ as a limit.

(b) If $g(x) = \sqrt{x^5 - 1}$, what is d^2g/dx^2 ?

(c) Sketch the parametric curve $x = t$, $y = \sqrt{t}$.

(d) Define an inflection point.

(e) Approximate $(1.9)^2$.

4. Multiple choice.

(a) The derivative of $1/(-x^2 + 2)^3$ is:

(i) $1/(-6x)(2 - x)^2$ (ii) $6x/(-x^2 + 2)^4$

(iii) $6x/(2 - x^2)^2$ (iv) $-2x/(-x^2 + 2)^4$

(b) If $x^2 + y^2 = 1$, then dy/dx is:

(i) $-y/x$ (ii) $-x/y$

(iii) $2x$ (iv) $2y$

(c) If $h'(v) > 0$, then $h(v)$ is

(i) increasing (ii) decreasing

(iii) positive (iv) negative

(d) $f(x) = x^3 + 2x + 1$ crosses the x-axis once in the interval:

(i) $(-2,-1)$ (ii) $(-1,0)$

(iii) $(0,1)$ (iv) $(1,2)$

(e) The mean value theorem states that:

(i) the average value of $f(a)$ and $f(b)$ is attained in (a,b) if f is continuous.

(ii) all continuous functions have a mean.

(iii) all differentiable functions have a mean.

(iv) the average slope of f in $[a,b]$ is in (a,b) if f is differentiable.

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5. Differentiation problems.

   (a) Find dy/dx if $y = (x^2 - 3)\sqrt{1/2 - x}$.

   (b) Find dy/dx if $y = (x^{2/3} + 2/3)/(x + \sqrt{x})$.

   (c) Find dy/dx if $y = 3t$ and $x = 4t^2 - t$.

   (d) Find dy/dx if $x^2y^{1/3} = xy$.

   (e) Find dy/dx if $y = f(g(x))$ where $f(x) = \sqrt{x^2 + x}$ and $g(x) = x^3 + x^{3/2}$.

6. Find the equation of the cubic polynomial $f(x)$ whose graph has an inflection point at $x = 2$ and a local maximum at $(0,5)$.

7. Antidifferentiation problems.

   (a) Evaluate $\int(\sqrt{t} + t^{-2} + 1)dt$.

   (b) Evaluate $\int\sqrt{5y/3 + 3/5}\, dy$.

   (c) Show that an antiderivative of $f(x) = (4x + 6)/(x^2 + 3x + 1)^2$ is $-2(x^2 + 3x + 1)^{-1} + 6$.

8. Water is being pumped from a square swimming pool with a length of 10 m. into a circular cylinder with radius 3 m. How fast is the water level rising in the cylinder if the water in the pool is dropping 10 cm./min. ?

9. (a) The difference between x and y is 10 . Find the minimum of xy .

   (b) Suppose x and y are related by $5x^2 + y = 5$. Find the maximum of x^2y .

10. At time t , a particle's position is given by $y = x^4/4 - 3x^2/2 + 4x$. Sketch the graph of the particle's velocity function.

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ANSWERS TO COMPREHENSIVE TEST

1. (a) True.

   (b) False; the derivative is $(6x + 1)(2x^4 + x) + (3x^2 + x)(8x^3 + 1)$.

   (c) False; π^3 is a constant whose derivative is zero.

   (d) True.

   (e) False; the tangent line is $u = (3z^2 - 4)\big|_2(z - 2) = 8(z - 2)$.

   (f) True.

   (g) True.

   (h) False; consider $f(x) = |x|$.

   (i) False; let $f(x) = g(x) = x$ on $[-1,0]$.

2. (a) Neither.

   (b) Vertical asymptote: $x = -2$; no horizontal asymptote.

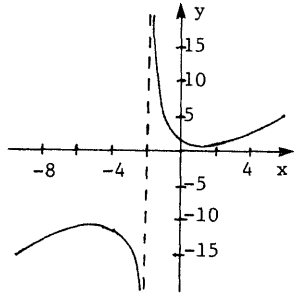
   (c) Nowhere.

   (d) Increasing: $(-\infty,-5)$ and $(1,\infty)$; decreasing: $(-5,-2)$ and $(-2,1)$.

   (e) Concave upward: $(-2,\infty)$; concave downward: $(-\infty,-2)$.

   (f) None.

   (g)



3. (a) $\lim_{\Delta x \to 0} \{[f(x + \Delta x) - f(x)]/\Delta x\}$

   (b) $5x^3(3x^5 - 8)/4(x^5 - 1)^{3/2}$

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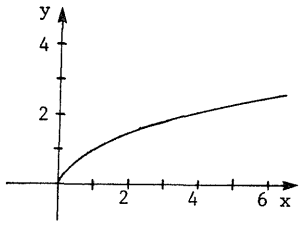
3 (continued)

(c)



(d) An inflection point is the point where concavity changes direction.

(e) 3.6

4. (a) ii

(b) ii

(c) i

(d) ii

(e) iv

5. (a) $(2x)\sqrt{1/2 - x} - (x^2 - 3)/2\sqrt{1/2 - x}$

(b) $[(2/3x^{1/3})(x + \sqrt{x}) - (x^{2/3} + 2/3)(1 + 1/2\sqrt{x})]/(x + \sqrt{x})^2$

(c) $3/(8t - 1)$

(d) $(2xy^{1/3} - y)/(x - x^2/3y^{2/3})$

(e) $[2(x^3 + x^{3/2}) + 1](3x^2 + 3\sqrt{x}/2)/2[(x^3 + x^{3/2})^2 + (x^3 + x^{3/2})]^{1/2}$

6. $f(x) = x^3/8 - 3x^2/4 + 5$

7. (a) $2t^{3/2}/3 - 1/t + t + C$

(b) $(2/5)(5y/3 + 3/5)^{3/2} + C$

(c) Differentiate $2(x^2 + 3x + 1)^{-1} + 6$ to get $f(x)$.

8. $(1000/9\pi)$ cm./min.

9. (a) -25

(b) 5/4

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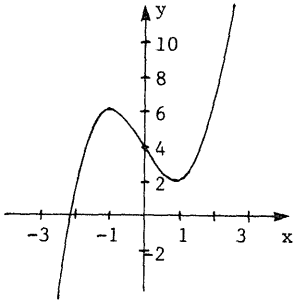
10.



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